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Perturbation Analyses for the Postbuckling and Imperfection Sensitivity of Circular Cylindrical Shells under Torsion

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1. Introduction

It is of great technical importance to clarify the whole aspect of the stability of circular cylindrical shells under torsion. To explore the problem systematically, the authors first treated the buckling problem and accurate solutions for the critical load and the wave number are obtained for a wide range of the shell geometry Z , under eight different boundary condition [1, 2]. Next, to clarify the postbuckling behavior, precise experimental studies were carried out, using polyester test cylinders with Z ranging from 20 to 1000 [3]. Then, the corresponding theoretical studies were performed, under the completely clamped boundary conditions[4]. It is found that both the results are in reasonable agreement. Later, the theoretical studies were extend to include the effect of imperfections, assuming the imperfections in the shape of the buckling mode [5].

In the foregoing postbuckling analyses, the Donnell equations were used and the problem was solved by directly applysing the Galerkin method. Another method of solution is to use the perturbation analyses based on the initial postbuckling theory of Koiter and Budiansky [6]. With this method, a number of imperfection sensitivity problems have been solved for various shell structures [6]. In particular, present problem has been solved by Budiansky [7], under three different boundary conditions. The senior author also treated the initial postbuckling problem of clamped cylindrical shells under compression [8]. In these studies, however, only the terms up to the second order expansion were used and the range of validity of the solution has not been examined. The main object of this study is to perform the perturbation analyses up to the sixth order and to clarify the range of applicability by comparing with the previous results obtained by direct nonlinear analyses.

2. Postbuckling Problem for Imperfect Cylindrical Shell

Assume that a circular cylindrical shell with mean radius R, length L, thickness h is subject to a torque T. (Fig.1). Denoting by $\bar{W}$ and W the initial and additional deflections, respectively, and by F the stress function, the basic equations are given by Donnell as follows:

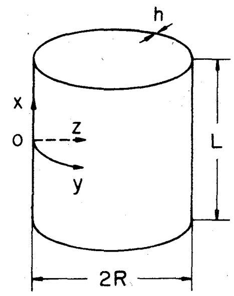

Fig. 1.

$$\nabla^4 F + Eh\,(R^{-1} W_{,xx} + W_{,xx} W_{,yy} - W_{,xy}^2$$

$$+ \bar{W}_{,xx} W_{,yy} - 2\,\bar{W}_{,xy} W_{,xy} + \bar{W}_{,yy} W_{,xx}) = 0 \qquad (1)$$

$$D\nabla^4 W - R^{-1} F_{,xx} - F_{,xx} (\bar{W} + W)_{,yy} + 2\,F_{,xy} (\bar{W} + W)_{,xy}$$

$$- F_{,yy} (\bar{W} + W)_{,xx} = 0 \qquad (2)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \qquad (3)$$

In the foregoing, D is the flexural rigidity, E and ν are Young's modulus and Poisson's ratio, respectively, while subscripts following a comma stand for differentiation. Stress resultants are expressed by F as

$$N_x = F_{,yy}\,, \quad N_y = F_{,xx}\,, \quad N_{xy} = -F_{,xy} \tag{4}$$

while in-plane displacements U and V are related to $\overline{W}$, W and F as

$$\left.\begin{aligned} &Eh(U_{,x} + \tfrac{1}{2}W_{,x}^2 + \overline{W}_{,x}W_{,x}) = F_{,yy} - \nu F_{,xx}\,,\\ &Eh(V_{,y} - R^{-1}W + \tfrac{1}{2}W_{,y}^2 + \overline{W}_{,y}W_{,y}) = F_{,xx} - \nu F_{,yy}\,,\\ &Eh(U_{,y} + V_{,x} + W_{,x}W_{,y} + \overline{W}_{,x}W_{,y} + \overline{W}_{,y}W_{,x}) = -2(1+\nu)F_{,xy} \end{aligned}\right\} \tag{5}$$

Assuming that both edges are rigidly clamped to stiff end plates, the boundary conditions are

$$x = \pm L/2 : \quad W = W_{,x} = 0 \tag{6a}$$

$$U_{,y} = V_{,y} = \int_0^{2\pi R} N_x\,dy = R\int_0^{2\pi R} N_{xy}\,dy - T = 0$$

or

$$F_{,xx} - \nu F_{,yy} = F_{,xxx} + (2+\nu)F_{,xyy} = [F_{,y}]_{y=0}^{2\pi R} = R[F_{,x}]_{y=0}^{2\pi R} + T = 0 \tag{6b}$$

The axial shortening δ and the angle of twist ψ are given by

$$\delta = -[U]_{x=-L/2}^{L/2}\,, \qquad \psi = R^{-1}[V]_{x=-L/2}^{L/2} \tag{7}$$

Here, we introduce the following notations for convenience:

$$\left.\begin{aligned} &\xi = \pi x/L, \quad \eta = \pi y/\ell, \quad \ell = \pi R/N, \quad c = 1/12(1-\nu^2),\\ &(u, v) = (L/\pi h^2)(U, V), \quad (\overline{w}, w) = (\overline{W}, W)/h, \quad f = F/Eh^3,\\ &\alpha = L^2/\pi^2 Rh, \quad \beta = (L/\pi R)N, \quad \tau = TL^2/2\pi^3R^2Eh^3,\\ &\overline{\delta} = R\delta/Lh, \quad \overline{\Psi} = R^2\psi/Lh, \quad Z = \sqrt{1-\nu^2}\,L^2/Rh, \quad k_s = \tau/c \end{aligned}\right\} \tag{8}$$

In the foregoing, N is the circumferential wave number, α , β and τ are non-dimensional factors relating to the shell geometry, wave number and applied torque, respectively. With these notations, the preceding equaions are rewritten as follows:

$$\bar{\nabla}^4 f + \alpha w_{,\xi\xi} + \beta^2 [\, w_{,\xi\xi} w_{,\eta\eta} - w_{,\xi\eta}^2 + \bar{w}_{,\xi\xi} w_{,\eta\eta} - 2\bar{w}_{,\xi\eta} w_{,\xi\eta} + \bar{w}_{,\eta\eta} w_{,\xi\xi} \,] = 0 \qquad (1)'$$

$$c\bar{\nabla}^4 w - \alpha f_{,\xi\xi} - \beta^2 [\, f_{,\xi\xi} (\bar{w} + w)_{,\eta\eta} - 2 f_{,\xi\eta} (\bar{w} + w)_{,\xi\eta} + f_{,\eta\eta} (\bar{w} + w)_{,\xi\xi} \,] = 0 \qquad (2)'$$

$$\bar{\nabla}^2 = \partial^2/\partial\xi^2 + \beta^2 \cdot \partial^2/\partial\eta^2 \qquad (3)'$$

$$\left.\begin{aligned} & u_{,\xi} + \tfrac{1}{2} w_{,\xi}^2 + \bar{w}_{,\xi} w_{,\xi} = \beta^2 f_{,\eta\eta} - \nu f_{,\xi\xi} , \\ & \beta v_{,\eta} - \alpha w + \beta^2 (\tfrac{1}{2} w_{,\eta}^2 + \bar{w}_{,\eta} w_{,\eta}) = f_{,\xi\xi} - \nu \beta^2 f_{,\eta\eta} , \\ & \beta u_{,\eta} + v_{,\xi} + \beta (w_{,\xi} w_{,\eta} + \bar{w}_{,\xi} w_{,\eta} + \bar{w}_{,\eta} w_{,\xi}) = -2(1+\nu)\beta f_{,\xi\eta} \end{aligned}\right\} \qquad (5)'$$

$$\xi = \pm \pi/2 : \qquad w = w_{,\xi} = 0 \qquad (6a)'$$

$$f_{,\xi\xi} - \nu\beta^2 f_{,\eta\eta} = f_{,\xi\xi\xi} + (2+\nu)\beta^2 f_{,\xi\eta\eta} = [\, f_{,\eta} \,]_{\eta=-\pi}^{\pi}$$
$$= (\beta/2\pi) [\, f_{,\xi} \,]_{\eta=-\pi}^{\pi} + \tau = 0 \qquad (6b)'$$

$$\bar{\delta} = -\frac{1}{\pi\alpha} [\, u \,]_{\xi=-\pi/2}^{\pi/2} , \quad \bar{\psi} = \frac{1}{\pi\alpha} [\, v \,]_{\xi=-\pi/2}^{\pi/2} \qquad (7)'$$

These are the governing equations for the postbuckling of clamped imperfect cylindrical shells under torsion.

3. Initial Postbuckling Problem for Perfect Cylindrical Shells

Putting $\bar{w} = 0$ in the preceding equations, the relevant basic equations become

$$\bar{\nabla}^4 f + \alpha w_{,\xi\xi} + \frac{1}{2}\beta^2(w;w) = 0 \qquad (1)''$$

$$L(w) \equiv c\,\bar{\nabla}^4 w - \alpha f_{,\xi\xi} - \beta^2(f;w) = 0 \qquad (2)''$$

$$\xi = \pm\pi/2 \;:\; M_i(w,f) = 0,\ (i = 1\sim 5),\quad M_6(w,f) = \tau \qquad (6)''$$

where

$$(f;w) = f_{,\xi\xi}w_{,\eta\eta} - 2f_{,\xi\eta}w_{,\xi\eta} + f_{,\eta\eta}w_{,\xi\xi} \qquad (9)$$

$$M_1(w,f) = w, \quad M_2(w,f) = w_{,\xi}, \quad M_3(w,f) = f_{,\xi\xi} - \nu\beta^2 f_{,\eta\eta},$$

$$M_4(w,f) = f_{,\xi\xi\xi} + (2+\nu)\beta^2 f_{,\xi\eta\eta}, \quad M_5(w,f) = [f_{,\eta}]_{\eta=-\pi}^{\pi},$$

$$M_6(w,f) = -(\beta/2\pi)\,[f_{,\xi}]_{\eta=-\pi}^{\pi} \qquad (10)$$

Letting w_0 and f_0 the solutions for the prebuckling axisymmetric state, we easily find that

$$w_0 = 0, \qquad f_0 = -(\tau/\beta)\,\xi\eta \qquad (11)$$

Denoting by w_1 and f_1 the small incremental deformation during buckling, the governing equations become

$$\left.\begin{aligned} &L_1(w_1, f_1) \equiv \bar{\nabla}^4 f_1 + \alpha w_{1,\xi\xi} = 0, \\ &L_2(w_1, f_1) \equiv c\,\bar{\nabla}^4 w_1 - \alpha f_{1,\xi\xi} - \beta^2(f_0^c; w_1) = 0, \\ &\xi = \pm\pi/2 \;:\; M_i(w_1, f_1) = 0, \quad (i = 1\sim 6) \end{aligned}\right\} \qquad (12)$$

With these equations, the critical load τ_c contained in the expression $f_0^c = -(\tau_c/\beta)\,\xi\eta$ can be determined as the lowest eigen-value for each prescribed wave number β . The corresponding eigen-functions w_1 and f_1 can be definitely determined with the condition $(w_1)_{max} = 1$.

Now we assume the solution in the buckled state can be expressed as

$$w = \sum_{n=1} \varepsilon^n w_n \ , \quad f = f_0(\tau) + \sum_{n=1} \varepsilon^n f_n \ , \quad \tau/\tau_c = 1 + \sum_{n=1} \varepsilon^n b_n$$
$$(n = 1, 2, 3 \cdots) \tag{13}$$

where

$$\left.\begin{aligned} f_0(\tau) &= f_0^{\,c} + (\tau - \tau_c)\dot{f}_0^{\,c} = f_0^{\,c} + \tau_c \dot{f}_0^{\,c} \sum_{n=1} \varepsilon^n b_n \ , \\ \dot{f}_0^{\,c} &= (\partial f_0/\partial \tau)_{\tau=\tau_c} = -(1/\beta)\xi\eta \end{aligned}\right\} \tag{14}$$

and where ε ($>$ 0) is a small parameter. Substituting these expressions into Eqs.(1)" and (2)" and equating to zero the terms with ε^n, we obtain the following set of the linear boundary value problems for w_n and f_n (n = 2, 3, 4, $\cdots$):

$$\left.\begin{aligned} & L_1(w_n, f_n) = \phi_n(\xi, \eta) \ , \qquad L_2(w_n, f_n) = \psi_n(\xi, \eta) \ , \\ & \xi = \pm \pi/2 \ : \ M_i(w_n, f_n) = 0, \quad (i = 1 \sim 6) \end{aligned}\right\} \tag{15}$$

where

$$\left.\begin{aligned} \phi_n(\xi, \eta) &= -\tfrac{1}{2}\beta^2 \sum_{k=1}^{n-1} (w_k \, ; \, w_{n-k}), \\ \psi_n(\xi, \eta) &= \beta^2 \sum_{k=1}^{n-1} [\, \tau_c b_k (\dot{f}_0^{\,c} ; w_{n-k}) + (f_k \, ; \, w_{n-k}) \,] \end{aligned}\right\} \tag{16}$$

Here, we note that Eqs.(12) are equivalent to the variational relation

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} [\, L_1(w_1, f_1)\,\delta f - L_2(w_1, f_1)\,\delta w \,] \, d\xi \, d\eta = 0,$$

from which we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} [L_1(w_n, f_n) f_1 - L_2(w_n, f_n) w_1] \, d\xi \, d\eta = 0,$$

$$(n = 2, 3, 4 \cdots) \tag{17}$$

These are solvability conditions for w_n and f_n . From Eqs.(16) and (17), we have

$$\sum_{k=1}^{n-1} [\tau_c b_k (f_0^c ; w_{n-k}, w_1) + \frac{1}{2}(f_1 ; w_k, w_{n-k}) + (f_k ; w_{n-k}, w_1)] = 0$$

$$(n = 2, 3, 4 \cdots) \tag{18}$$

where

$$(f_\ell ; w_m, w_n) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} (f_\ell ; w_m) w_n \, d\xi \, d\eta \tag{19}$$

Equations (18) serve to determine b_{n-1} ($n \geqq 2$) in terms of w_m, f_m and b_m with $m \leqq n-1$.

In addition, it is to be noted that the homogeneous problems corresponding to Eqs.(15) are identical with Eqs.(12) which have nontrivial solutions w_1 and f_1 . Hence, the general solution of Eqs.(15) will be expressed as

$$w_n = \hat{w}_n + c_n w_1, \qquad f_n = \hat{f}_n + c_n f_1 \tag{20}$$

where $\hat{w}_n$ and $\hat{f}_n$ are a pair of particular solutions while c_n is an arbitrary constant. To determine c_n , the following three kinds of conditions are considered:

Condition (a) requires the integral of (w_n)^2 to be minimum. That is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} (w_n)^2 \, d\xi \, d\eta = min. \tag{21a}$$

Condition (b) requires the residues ϕ_{n+1} and ψ_{n+1} to be orthogonal to f_0^c and w_0^c (= 0). That is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} f_{n+1} f_0^c \, d\xi \, d\eta = 0 \tag{21b}$$

Condition (c) is an application of the Galerkin method. That is, first assume w as

$$w = \varepsilon w_1 + \varepsilon^2 w_2 + \cdots + \varepsilon^n (\hat{w}_n + C_n w_1)$$

and determine f with Eqs.(1)" and (6)". Then, considering the terms up to the order ε^{n+2}, apply the Galerkin condition to Eq.(2)". That is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} L(w)\, w_1 \, d\xi \, d\eta = 0 \tag{21c}$$

With Eqs.(15), (18) together with (20), we can determine b_1 , w_2 , f_2 , b_2 , w_3 , f_3 , b_3 $\cdots$ successively. In each case, it is found that

$$b_{2i-1} = C_{2i} = 0, \quad (i = 1, 2, 3 \cdots) \tag{22}$$

Once w_n , f_n and b_n are found, the connections of $\bar{\psi}$ and $\bar{\delta}$ with τ can be determined, considering Eqs.(7)' and (5)'.

4. Effect of Imperfections in the Shape of the Buckling Mode

In this case, the initial deflection will be expressed as $\bar{w} = \mu w_1$ ($\mu \geqq 0$), where μ is the imperfection amplitude. Hence, the governing equations become as

$$\left.\begin{aligned} L_3(w,f) &\equiv \bar{\nabla}^4 f + \alpha w_{,\xi\xi} + \tfrac{1}{2}\beta^2(w;w) + \mu\beta^2(w_1;w) = 0, \\ L_4(w,f) &\equiv c\,\bar{\nabla}^4 w - \alpha f_{,\xi\xi} - \beta^2(f;w) - \mu\beta^2(f;w_1) = 0, \end{aligned}\right\} \tag{23}$$

$$\xi = \pm\pi/2 \; : \; M_i(w,f) = 0, \; (i = 1 \sim 5), \; M_6(w,f) = \tau \tag{6''}$$

To obtain approximate solutions utilizing the preceding asymptotic solutions, we assume w and f as

$$w = \sum_{n=1} \gamma^n w_n, \quad f = f_0^c + (\tau - \tau_c)\dot{f}_0^c + \sum_{n=1} \gamma^n f_n, \quad (n = 1, 2, 3 \cdots) \quad (24)$$

and apply the Galerkin condition for the determination of γ , which leads to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \left[L_3(w, f) \frac{\partial f}{\partial \gamma} - L_4(w, f) \frac{\partial w}{\partial \gamma} \right] d\xi d\eta = 0 \qquad (25)$$

Considering w_n , f_n and b_n up to n = 6 and performing the integration, we finally obtain

$$\begin{aligned}
&\left(1 - \frac{\tau}{\tau_c}\right) \gamma (1 + G_1 \gamma^2 + G_2 \gamma^4) + b_2 \gamma^3 (1 + G_1 \gamma^2 + G_2 \gamma^4) \\
&+ b_4 \gamma^5 (1 + G_1 \gamma^2) + b_6 \gamma^7 - \mu \left[\frac{\tau}{\tau_c} (1 + H_1 \gamma^2 + H_2 \gamma^4) \right. \\
&\left. + I_1 \gamma^2 + I_2 \gamma^4 + I_3 \gamma^6 \right] = 0 \qquad (26)
\end{aligned}$$

where

$$\begin{aligned}
&A = (\dot{f}_0^c; w_1, w_1), \qquad G_1 = (2/A)[(\dot{f}_0^c; w_2, w_2) + 2(\dot{f}_0^c; w_1, w_3)], \\
&G_2 = (3/A)[(\dot{f}_0^c; w_3, w_3) + 2(\dot{f}_0^c; w_1, w_5) + 2(\dot{f}_0^c; w_2, w_4)], \\
&H_1 = (3/A)(\dot{f}_0^c; w_1, w_3), \qquad H_2 = (5/A)(\dot{f}_0^c; w_1, w_5), \\
&I_1 = (3/\tau_c A)[(f_1; w_1, w_2) + (f_2; w_1, w_1)], \\
&I_2 = (5/\tau_c A)[(f_1; w_1, w_4) + (f_2; w_1, w_3) + (f_3; w_1, w_2) + (f_4; w_1, w_1)], \\
&I_3 = (7/\tau_c A)[(f_1; w_1, w_6) + (f_2; w_1, w_5) + (f_3; w_1, w_4) + (f_4; w_1, w_3) \\
&\qquad + (f_5; w_1, w_2) + (f_6; w_1, w_1)] \qquad (27)
\end{aligned}$$

Equation (26) gives the relation between the load τ and deformation γ for each prescrobed value of the imperfection amplitude μ . When b_2 is negative, the cylinder will become imperfection sensitive and the snap-through critical load τ_s is given by the peak point along τ - γ relation. It is found that with the increase in μ, the peak point finally disappears and degenerates into the inflection point. Hence, the critical load τ_s for these values of μ is defined by the load corresponding to the inflection point along the τ - γ relation.

In case when the perturbation terms up to the fourth order are considered, Eq.(26) becomes

$$\left(1-\frac{\tau}{\tau_c}\right)\gamma\left(1+G_1\gamma^2\right)+b_2\gamma^3\left(1+G_1\gamma^2\right)+b_4\gamma^5 \\ -\mu\left[\frac{\tau}{\tau_c}\left(1+H_1\gamma^2\right)+I_1\gamma^2+I_2\gamma^4\right]=0 \tag{28}$$

while considering only the second order terms as usual, we have

$$\left(1-\frac{\tau}{\tau_c}\right)\gamma+b_2\gamma^3-\mu\left(\frac{\tau}{\tau_c}+I_1\gamma^2\right)=0 \tag{29}$$

for which the condition for τ_s becomes

$$\tau_s/\tau_c=1-2\mu I_1\gamma+3b_2\gamma^2 \tag{30-Y}$$

Neglecting $I_1\gamma^2$ in Eq.(29), we have

$$\left(1-\frac{\tau_s}{\tau_c}\right)^{3/2}=\frac{3}{2}\sqrt{-3b_2}\,\mu\cdot\frac{\tau_s}{\tau_c} \tag{30-K}$$

while with further assumption that $\mu\cdot(\tau/\tau_c)\cong\mu$, we have

$$\frac{\tau_s}{\tau_c}=1-1.89\left(-b_2\right)^{1/3}\mu^{2/3} \tag{30-H}$$

Equations (30-K) and (30-H) correspond to the well-known expressions obtained by Koiter [6] and Hutchinson [9], respectively.

5. Method of Solution

In the previous paper [4], the postbuckling of clamped perfect cylindrical shells under torsion has been solved by directly applying the Galerkin method to the nonlinear basic equations (1)" and (2)". That is, the solution w is first assumed as

$$w = \sum_{m=1}\sum_{n=0} a_{mn} w_{mn} = \sum_{m=1}\sum_{n=0} a_{mn}(\Psi_{m-1,n} + \Psi_{m+1,n}),$$

$$(m = 1, 2, 3 \cdots, \quad n = 0, 1, 2 \cdots) \tag{31}$$

where

$$\Psi_{pq} = \cos(p\xi + q\eta) + (-1)^p \cos(p\xi - q\eta) \tag{32}$$

and where a_{mn} are unknown parameters. Substituting the foregoing expression into Eq.(1)" and integrating, we obtain the stress function f satisfying the boundary conditions (6)" exactly. With these expressions for w and f, we apply the Galerkin method to Eq.(2)", which leads to the conditions

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} L(w)\, w_{rs}\, d\xi\, d\eta = 0, \quad (r = 1, 2, 3 \cdots, \; s = 0, 1, 2 \cdots) \tag{33}$$

When the values for Poisson's ratio ν, shell geometry α (or Z), wave number β (or N) and applied load τ (or k_s) are given, Eqs.(33) give a set of cubic equations in a_{mn}, which can be solved for a_{mn} with the Newton-Raphson method. Once a_{mn} are determined, w and f, consequently $\bar{\delta}$ and $\bar{\psi}$, can be determined and the problem will be solved.

In the foregoing, detailed expressions are omitted to save the space. With the use of these expressions, solutions for the present perturbation analyses can be obtained without difficulty, since the governing equations for the perturbation terms correspond to the special cases for the original basic equations (1)" and (2)".

Solutions for the prebuckling state, w_0 and f_0, have been given by Eqs.(11). To obtain the solutions for w_n and f_n, we first assume w_n as follows:

$$w_1 = \sum_m a_{m1}^{(1)} w_{m1}, \qquad w_2 = \sum_m \left(a_{m0}^{(2)} w_{m0} + a_{m2}^{(2)} w_{m2}\right),$$

$$w_3 = \sum_m \left(a_{m1}^{(3)} w_{m1} + a_{m3}^{(3)} w_{m3}\right),$$

$$w_4 = \sum_m \left(a_{m0}^{(4)} w_{m0} + a_{m2}^{(4)} w_{m2} + a_{m4}^{(4)} w_{m4}\right),$$

$$w_5 = \sum_m \left(a_{m1}^{(5)} w_{m1} + a_{m3}^{(5)} w_{m3}\right),$$

$$w_6 = \sum_m \left(a_{m0}^{(6)} w_{m0} + a_{m2}^{(6)} w_{m2} + a_{m4}^{(6)} w_{m4}\right),$$

$$(m = 1, 2, 3 \cdots) \qquad (34)$$

Applying the similar procedure as stated in the foregoing to the relevant basic equations, we will obtain the expression for the corresponding stress function f_n and then equations for the determination of the parameter $a_{mn}^{(i)}$. When $i = n = 1$, the latter equations represent a set of homogeneous linear equations and the condition for the existance of non-trivial solutions gives the critical load τ_c as well as the eigen-functions w_1 and f_1. For the other cases, we will have simultaneous linear equations for $a_{mn}^{(i)}$, from which we can obtain a pair of particular solutions $\hat{w}_n$ and $\hat{f}_n$ ($n \geqq 2$). Then, applying one of the conditions (a), (b) and (c) stated in Eqs.(21), we will find the corresponding constant C_n, which gives the required solutions for w_n and f_n. Once w_n and f_n ($n \geqq 1$) are determined, we can find the load coefficients b_n ($n \geqq 1$) with successive applications of the solvability condition (18). In this way, we can determine w_1 , f_1 , b_1 , w_2 , f_2 , b_2 $\cdots$, stepwisely. Thus, recalling that b_{2i-1} (i = 1, 2, 3 $\cdots$) = 0, we will obtain the sixth order solution as

$$\left.\begin{aligned} w &= \sum_{n=1}^{6} \varepsilon^n w_n , \qquad f = f_0^c + (\tau - \tau_c)\dot{f}_0^c + \sum_{n=1}^{6} \varepsilon^n f_n , \\ \tau/\tau_c &= 1 + \varepsilon^2 b_2 + \varepsilon^4 b_4 + \varepsilon^6 b_6 \end{aligned}\right\} \quad (35)$$

The second and the fourth order solutions will be given by retaining n up to 2 and 4, respectively. Once w_n , f_n and b_n are found, we can easily determine $\bar{\delta}$ and $\bar{\psi}$ as well as the values of the coefficients appearing in the asymptotic expressions for the imperfection sensitivity, i.e., Eqs.(27).

6. Numerical Results and Discussions

6-1. Postbuckling Behavior of Perfect Cylindrical Shells

To compare with the previous results [4], calculations are carried out for the cases when Z are 20, 50, 100, 200, 500 and 1000, by taking ν =0.3 and R/h = 405. The wave numbers N in each case are also taken to be the same as the those previously treated. For each case, perturbation terms up to the sixth order are determined with the three conditions (a), (b) and (c) for C_n . Practically accurate solutions were obtained by retaining the coefficient $a_{mn}^{(i)}$ (i = 1~6) with m ranging from 1 to 20.

Comparison of the present and previous solutions for the typical postbuckling behaviors are shown in Figs.2 to 4, for the cases when Z are 20, 100 and 500, respectively. In the figures, thick solid line corresponds to the previous solution, while the curves labeled with 2, 4b, for instance, designate the present solutions of the second order as well as those of the

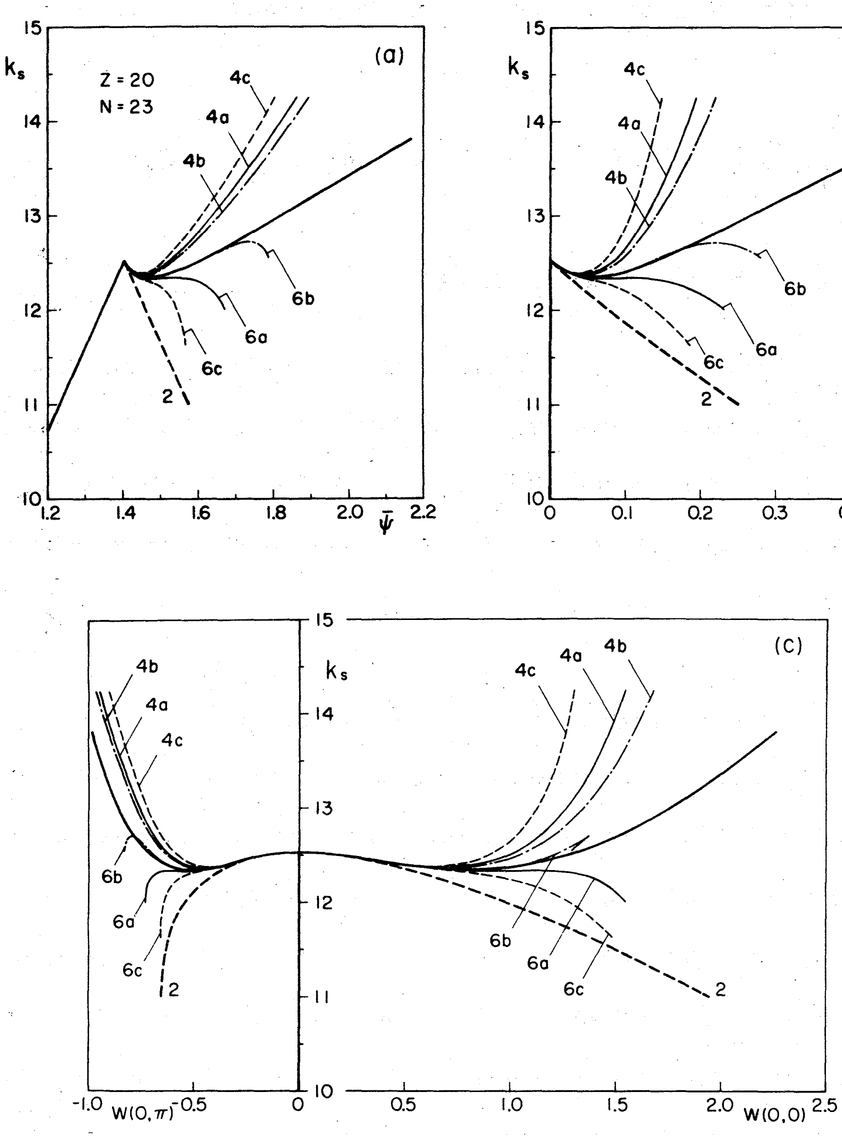

Fig.2. Comparison of the typical postbuckling behaviors for the clamped cylindrical shell with $Z = 20$, $R/h = 405$, $N = 23$.
(a) Relation between the torque k_s and the angle of twist $\bar{\psi}$.
(b) Relation between the torque k_s and the axial shortening $\bar{\delta}$.
(c) Relation between the torque k_s and the maximum inward and outward deflections, $w(0, 0)$ and $w(0, \pi)$.

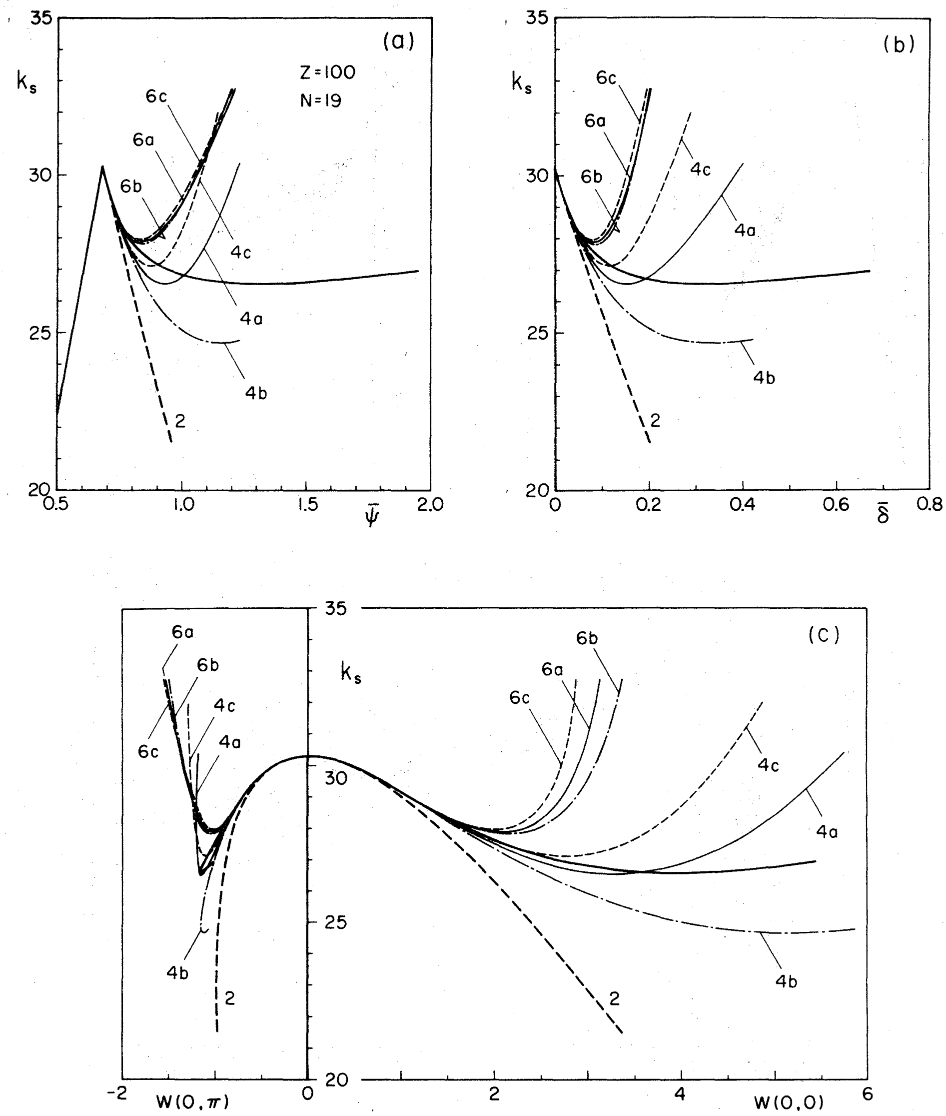

Fig.3. Comparison of the typical postbuckling behaviors for the clamped cylindrical shell with Z = 100, R/h = 405, N = 19.
(a) Relation between the torque k_s and the angle of twist $\bar{\psi}$.
(b) Relation between the torque k_s and the axial shortening $\bar{\delta}$.
(c) Relation between the torque k_s and the maximum inward and outward deflections, $w(0, 0)$ and $w(0, \pi)$.

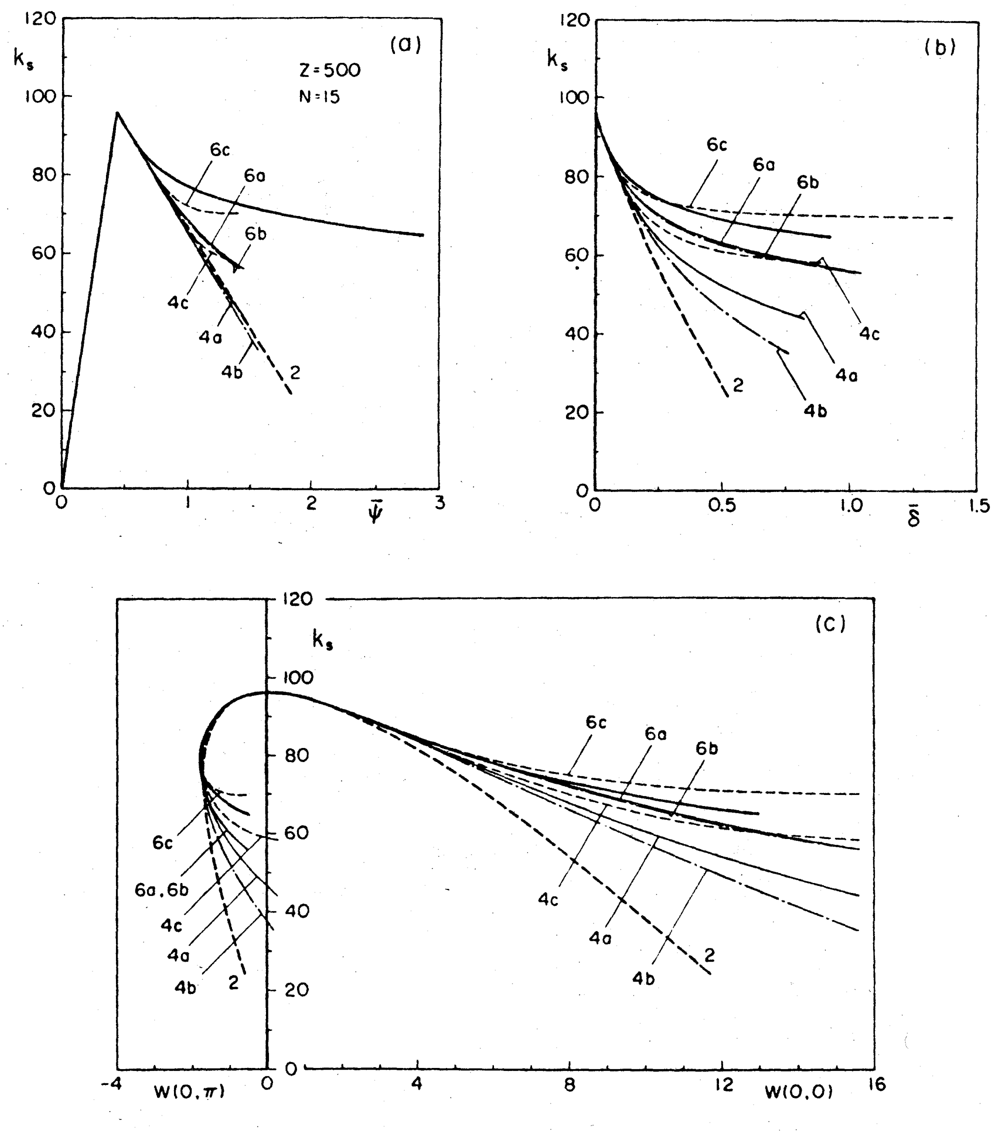

Fig.4. Comparison of the typical postbuckling behaviors for the clamped cylindrical shell with Z = 500, R/h = 405, N = 15.

(a) Relation between the torque k_s and the angle of twist $\bar{\psi}$.

(b) Relation between the torque k_s and the axial shortening $\bar{\delta}$.

(c) Relation between the torque k_s and the maximum inward and outward deflections, $w(0, 0)$ and $w(0, \pi)$.

fourth order associated with the condition (b), respectively. From these results, the following conclusions may be obtained:

(1) The second order solution is valid only in the immediate vicinity of the critical state. It will lead to a serious error for short shells as it predicts monotonous decreases of the load after buckling.

(2) The effect of the minimum postbuckling load is duly considered in the higher order solutions.

(3) The range of validity of the sixth order solution is somewhat greater than that of the fourth order solution but the improvement is not so good to compensate the increased effort in the analyses. Hence, it seems that the present perturbation analyses cannot be used as substitute for the direct nonlinear analyses.

(4) No significant differences are seen among the three conditions concerning the convergency of the resulting solutions, although conditions (a), (b), (c) seem to lead to the most favourable results for the cases when Z are 100, 20 and 500, respectively. Hence, the condition (a) should be recommended from its clear physical meaning as well as the ease in the application.

6-2. Imperfection Sensitivity

In the previous paper [5], the effect of imperfections in the shape of the buckling mode is treated with the direct nonlinear analyses, for the cylinders with the same values of Z as treated in [4]. For example, some of the results obtained for Z = 100 are reproduced in Figs.5a and 5b, where μ is the imperfection amplitude defined previously. Further, small circles denote the snap-though critical load k_{ss} corresponding to the peak point while those with vertical and lateral bars designate the critical load corresponding to the inflection point along the $k_s \sim \bar{\psi}$ and $k_s - w(0, 0)$

curves, respectively.

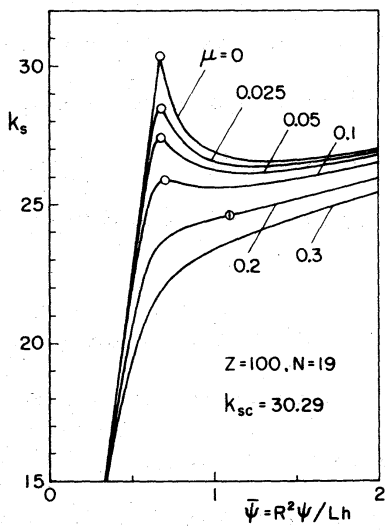

Fig.5a. Previous results for the effect of imperfections on the relation between k_s and $\bar{\psi}$: Z = 100, N = 19.

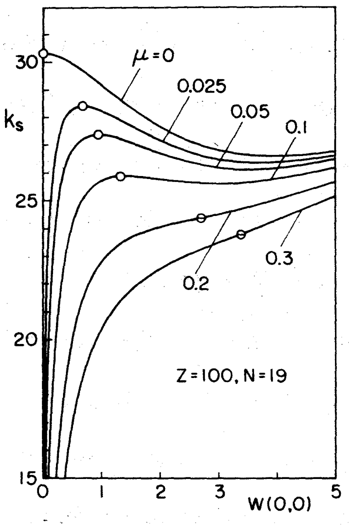

Fig.5b. Previous results for the effect of imperfections on the relation between k_s and $w(0,0)$: Z = 100, N = 19.

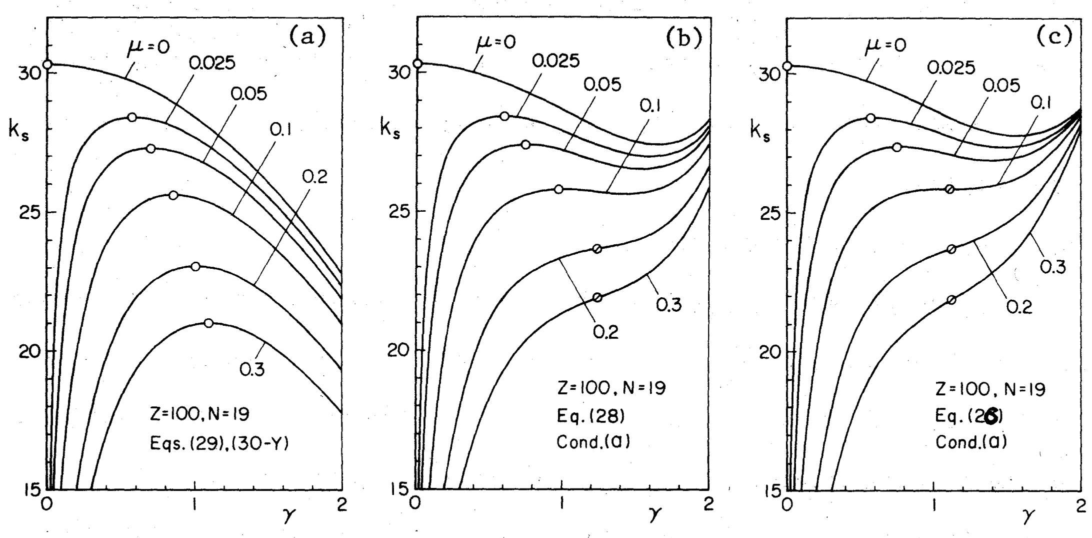

Fig.6. Effect of the imperfection amplitude μ on the relation between k_s and γ : Z = 100, N = 19.

(a) Second order expression, (30-Y),

(b) Fourth order expression, (28), Condition (a),

(c) Sixth order expression, (26), Condition (a).

To compare with the foregoing results, effects of μ on the relation between k_s and γ are determined with the present asymptotic analyses based on the second, fourth and sixth order expressions, which are shown in Figs.6(a) to 6(c), respectively. In the figures, the critical load defined by the inflection point is marked with small circles with a oblique bar. It is to be noted that no inflection point appears in the second order expression, in contrast to the higher order expressions.

Similar calculations are carried out for the remaining cases with Z different from 100. From these, the effect of μ on the critical load ratio k_{ss} / k_{sc} is determined for each case, with the results as shown in Figs.7(a) through 7(f). In these figures, the symbols O , Φ and θ denote the previous results obtained by the direct nonlinear analyses [5], while curves labeled with (Y), (K) and (H) stand for the present results based on the second order expressions, (30-Y), (30-K) and (30-H), respectively. From these results, the following conclusions may be obtained:

(1) For sufficiently small values of μ , the asymptotic estimates for the critical load coincide with the accurate ones, as they should.

(2) Denoting by μ_c the upper bound of μ for which the shell buckles with snap-through phenomena, it was found in the previous paper [5] that μ_c = 0.007, 0.05, 0.15, 0.35, 1.1 and 2.0 for the cases when Z are 20, 50, 100, 200, 500 and 1000, respectively. In general, the asymptotic predictions for the critical load are valid so far as μ is less than μ_c .

(3) Among the second order estimates, the Koiter's expression always gives the highest value for the critical load, which may lead to some overestimation for long shells with Z greater than 200. On the contrary, the Hutchinson's expression always underestimates the critical load while the predictions based on the authors' expression, (30-Y), almost coincide with

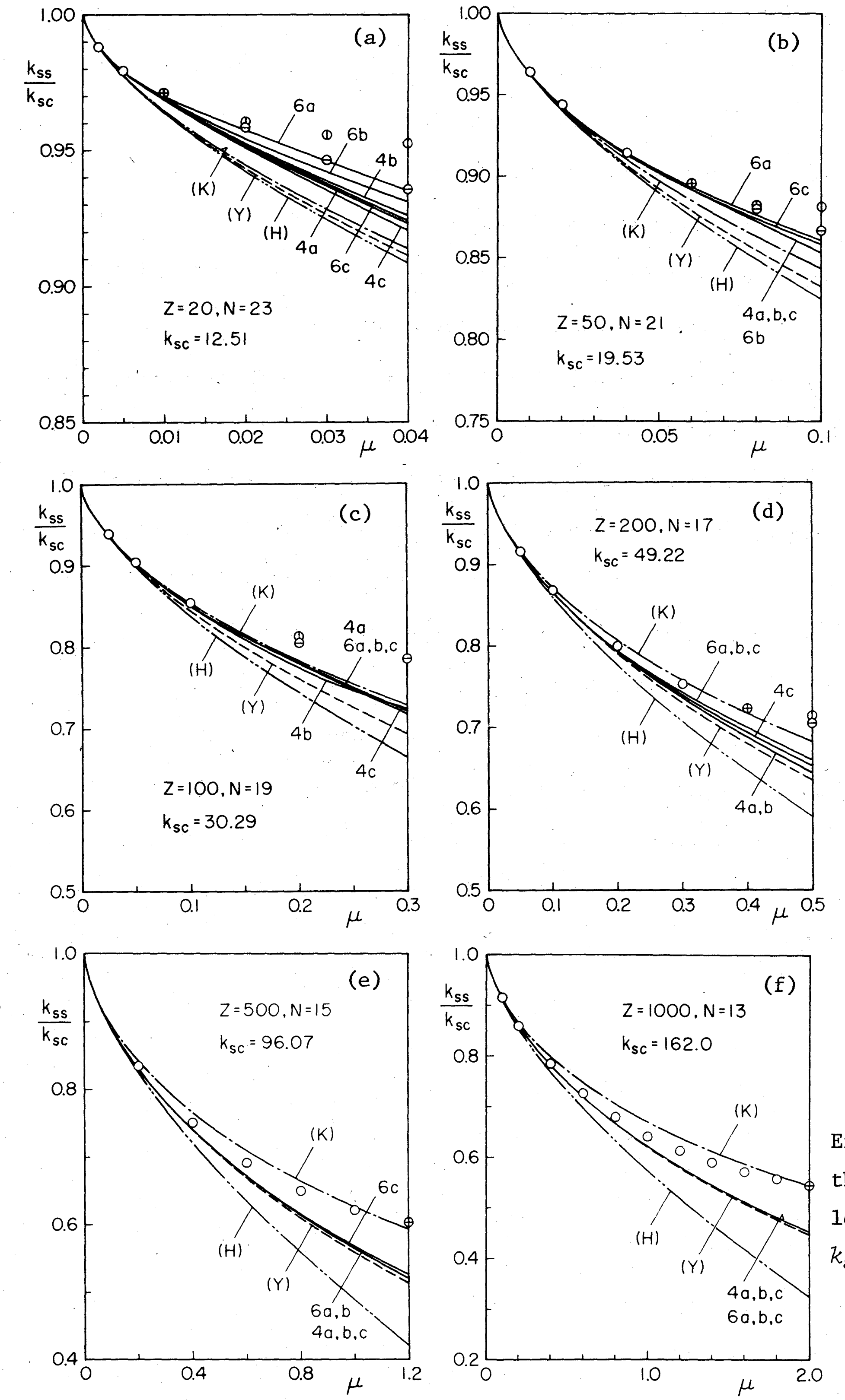

Fig. 7. Effect of μ on the critical load ratio k_{ss}/k_{sc}

those of the higher order expressions for long shells with Z greater than 200. In any case, the second order expression may lead to a serious error since we cannot estimate μ_c , that is, the proper range of validity for μ .

(4) There are no significant differences among the predictions based on the fourth and sixth roder expressions as well as those associated with the conditions (a), (b) and (c).

(5) The fourth order expression associated with the condition (a) will be of great technical importance since we can estimate the range of applicability through approximate determination of μ_c .

Finally, values of the coefficients appearing in Eq.(28), associated with the condition (a), are given in Table 1 for completeness.

Table 1. Values of the coefficients appearing in Eq.(28) : R/h = 405, ν = 0.3, Condition (a). (E-2 = 10^{-2})

Z	20	50	100	200	500	1000
N	23	21	19	17	15	13
k_{sc}	12.51	19.53	30.29	49.22	96.07	162.0
τ_c	1.146	1.789	2.774	4.507	8.798	14.83
b_2	-7.132E-2	-7.987E-2	-6.181E-2	-4.039E-2	-1.974E-2	-1.128E-2
b_4	1.117E-1	3.262E-2	7.694E-3	1.313E-3	9.535E-5	1.974E-5
G_1	-1.241E-1	-1.257E-1	-8.248E-2	-4.467E-2	-1.606E-2	-7.731E-3
H_1	-1.080E-1	-9.900E-2	-6.481E-2	-3.538E-2	-1.269E-2	-6.207E-3
I_1	1.094E-1	1.588E-1	1.293E-1	8.629E-2	4.294E-2	2.460E-2
I_2	-5.095E-1	-1.713E-1	-4.918E-2	-1.136E-2	-1.525E-3	-3.888E-4

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