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On the fixed point set of a unipotent transformation on generalized flag varieties

Naohisa Shimomura
Hiroshima University

Introduction

Let $G = GL_n$ be the general linear group defined over a field K. Let P be a parabolic subgroup of G. For a unipotent element u of G, put

$$(G/P)_u = \{gP \in G/P \mid u \cdot gP = gP\},$$

the fixed point subvariety of u in a generalized flag variety G/P. The author [2] obtained a locally closed partition of $(G/P)_u$ into affine spaces. This is a generalization of a result of N. Spaltenstein [3]. The purpose of this report is to give an alternate proof to the result of [2]. The proof in this report is simpler than that of [2] and, it seems, applicable for other groups. Some applications (in particular, on the character theory of the finite general linear groups) of this paper are described in [1] with other results on the Springer representations of Weyl groups for reductive groups.

Notations. Let V be a vector space over a field K. If $\{x_\nu \mid \nu \in I\}$ is a subset of V, then we denote by $\langle x_\nu \mid \nu \in I \rangle$ the subspace spanned by $\{x_\nu\}$. We denote by $\mathbb{N}$ the set of all natural numbers. For $n \in \mathbb{N}$, let $\mathbb{A}^n$ be the n-dimensional affine space over K. If $\{X_\nu\}$ is a family of subsets of a set X, then $X = \coprod_\nu X_\nu$ means the direct sum decomposition of X. A partition λ of n means a sequence $\lambda = (n_1, n_2, \cdots, n_r)$ such that $n_i \in \mathbb{N}$ $(i=1,\cdots,r)$, $n_1+n_2+\cdots+n_r = n$ and $n_1 \geq n_2 \geq \cdots \geq n_r > 0$.

§1. Preliminaries

Let G, P and u be as in the introduction. There exists $\mu = (\mu_1,\cdots,\mu_r)$ (resp. $\lambda = (\lambda_1,\cdots,\lambda_s)$), a partition of n, such that P (resp. u) is conjugate to P_μ (resp. u_λ), where P_μ is a parabolic subgroup of G whose Levi subgroup is isomorphic to $\prod_{i=1}^{r} GL_{\mu_i}$ (resp. the unipotent element of Jordan type diag $(J_1,\cdots,J_s)$, $J_i = \underbrace{\begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}}_{\lambda_i} \Big\} \lambda_i$). Then $(G/P)_u$ is isomorphic to $(G/P_\mu)_{u_\lambda}$.

For λ, a partition of n, we can associate the Young diagram of type λ, in the usual way.

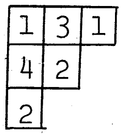
Definition 1. Let λ and μ be partitions of n. Put $\mu = (\mu_1,\cdots,\mu_r)$

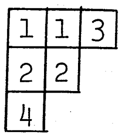
(1) A μ-tableau of type λ is a Young diagram of type λ whose nodes are numbered with the figures from 1 to r such that the cardinality of the nodes with figure i is μ_i.

(2) A μ-tableau is said to be semi-standard if, in each row,

the sequence of the figures on the nodes increases (may be stationary).

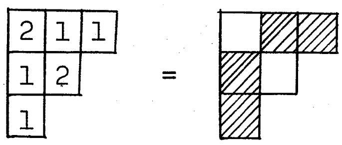
Example. If $\lambda = (3, 2, 1)$ and $\mu = (2, 2, 1, 1)$, then

(1) 

(2) 

are μ-tableaus of type λ ((2) is semi-standard). If $\mu = (\mu_1, \mu_2)$, then, for simplicity, we may write $\boxed{1}$ as ▨ and $\boxed{2}$ as □.

Example. If $\lambda = (3, 2, 1)$ and $\mu = (4, 2)$, then



Let $\widetilde{L_\mu(\lambda)}$ (resp. $L_\mu(\lambda)$) be the set of all μ-tableaus of type λ (resp. the set of all semi-standard μ-tableaus of type λ).

§2. The Grassmann manifold

Let $V = \langle v_1, \cdots, v_n \rangle$ be an n-dimensional vector space over a field K with basis $\{v_1, \cdots, v_n\}$. We denote by $G_k(V)$ the Grassmann manifold defined by the set of all k-dimensional subspaces of V. Put $L_k = \{(s_1, \cdots, s_k) \in \mathbb{N}^k \mid 1 \le s_1 < s_2 < \cdots < s_k \le n\}$, a set of increasing sequence of natural numbers. For $s = (s_1, \cdots, s_k) \in L_k$, let S_s be the set of vector subspaces defined by

$$\{\langle v_{s_m} + \sum_{i > s_m} a_{mi} v_i \mid 1 \le m \le k \rangle \mid a_{mi} \in K\}.$$

We remark that we can associate to S_s the following tableau :

v_1	v_2	$\cdots$	v_{s_1}	$\cdots$	v_{s_m}	$\cdots$	v_{s_k}	$\cdots$	v_n	.

The next lemma gives a well-known cellular decomposition of the Grassmann manifold.

<u>Lemma</u> 1. (1) $G_k(V) = \coprod_{s\in L_k} S_s$,

(2) $\langle v_{s_m} + \sum_{i>s_m} a_{mi}v_i \mid 1 \le m \le k\rangle = \langle v_{s_m} + \sum_{I_m} a'_{mi}v_i \mid 1 \le m \le k\rangle$,

<u>where</u> I_m <u>is a condition</u> : $i > s_m$, $i \neq s_{m+1},\cdots,s_k$,

(3) <u>put</u> $e(s) = \sum_{m=1}^{k}\{(n-s_m)-(k-m)\}$, <u>then by</u> (2), <u>we have an isomorphism</u> $\mathbb{A}^{e(s)} \xrightarrow{\sim} S_s$ <u>under a mapping</u> : $(\cdots,a_{mi},\cdots) \mapsto \langle v_{s_m} + \sum_{I_m} a_{mi}v_i \mid 1 \le m \le k\rangle$,

(4) S_s <u>is a locally closed subset of</u> $G_k(V)$ <u>in the</u> K-<u>Zariski topology</u>.

Let N be a nilpotent transformation of V. We take a Jordan basis $\{w_{ij_i} \mid 1 \le j_i \le \ell_i\}$ of V satisfying the following requirement :

$$\ell_1 \le \ell_2 \le \cdots \le \ell_d, \qquad Nw_{ij} = w_{i+1\,j} \qquad \text{and} \qquad Nw_{dj} = 0.$$

We remark that this basis forms a Young diagram of degree n and of type $\lambda = \lambda(N) = (\underbrace{d,\cdots,d}_{\ell_1},\cdots,\underbrace{1,\cdots,1}_{\ell_d-\ell_{d-1}})$.

<u>Example</u>. Let $\dim V = 8$. If N has two Jordan blocks of dimension 3 and one Jordan block of dimension 2, then

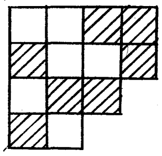
w_{31}	w_{21}	w_{11}
w_{32}	w_{22}	w_{12}
w_{33}	w_{23}	

.

Put $u_\lambda = 1_n + N$, 1_n is the identity matrix of size n, then u_λ is a unipotent element of $GL_n = GL(V)$ of Jordan type λ. We place w_{ij} in the following way :

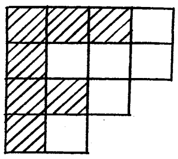
$$v_1 = w_{1\ell_1}, \cdots, v_{\ell_1} = w_{11}, v_{\ell_1+1} = w_{2\ell_2}, \cdots, v_{\ell_1+\ell_2} = w_{21}, \cdots, v_n = w_{d1}.$$

For $\bar{k} = (k, n-k)$ and $\lambda = \lambda(N)$, put

$\widetilde{L_{\bar{k}}(\lambda)} = \{\bar{k}\text{-tableaus of type } \lambda\} = \left\{\right.$  $\left|\right.$ the number of ▨ is k $\left.\right\}$.

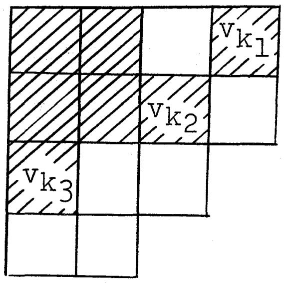
We have a bijective correspondence between $\widetilde{L_{\bar{k}}(\lambda)}$ and L_k by making a sequence $(s_1, \cdots, s_k) \in L_k$ if v_{s_i} is in a node ▨. Then by Lemma 1, (1), we can write $G_k(V) = \coprod_{\ell \in \widetilde{L_{\bar{k}}(\lambda)}} S_\ell$. Put

$$G_k(V)^N = \{W \in G_k(V) \mid N(W) \subset W\},$$
$$S_\ell^N = S_\ell \cap G_k(V)^N.$$

Let $L_{\bar{k}}(\lambda)$ be the set of all semi-standard $\bar{k}$-tableau of type $\lambda = \lambda(N)$, e.g. $L_{\bar{k}}(\lambda) = \left\{\right.$  $\left.\right\}$.

<u>Lemma</u> 2. <u>Let</u> $\ell \in \widetilde{L_{\bar{k}}(\lambda)}$. <u>In order to have</u> $S_\ell^N \neq \phi$, <u>it is necessary and sufficient that</u> $\ell \in L_{\bar{k}}(\lambda)$

Proof. We assume $S_\ell^N \neq \phi$. For this $\ell \in \widetilde{L_{\bar{k}}(\lambda)}$, let $v_{k_m} = w_{i_m j_m}$ $(m = 1, 2, \cdots ;\ k_1 < k_2 < \cdots)$ be the w_{ij} which is in the rightest node in the m-th row from the top in the tableau obtained by extracting the nodes □ from ℓ. For example

$\ell =$ 

.

For $W \in S_\ell^N$, there exists $a_{mj} \in K$ such that

$$v_{k_m} + \sum_{j>k_m} a_{mj} v_j \in W \quad (m=1,2,\cdots).$$

By $N(W) \subset W$, we have

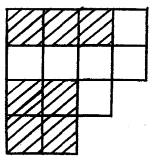
$$N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \in W \quad (0 \le h_m \le d-i_m).$$

The set $\{N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \mid {m=1,2,\cdots, \atop 0\le h_m \le d-i_m}\}$ is linearly independent and the number of its elements is greater than $k = \dim W$. Hence the set

$$\left\{N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \;\middle|\; {m=1,2,\cdots \atop 0\le h_m \le d-i_m}\right\}$$

must be a basis of W, which implies, by definition,

$$\ell \in L_{\overline{k}}(\lambda).$$

Conversely, for a semi-standard $\ell =$  $\in L_{\overline{k}}(\lambda)$, put

$W = \langle w_{ij}$ in ▨$\rangle$. Then $W \in S_\ell^N$. This means that $S_\ell^N \neq \phi$. The proof of the lemma is thus completed.

Let $\ell \in L_{\overline{k}}(\lambda)$. In the tableau ℓ, let $v_{k_m} = w_{i_m j_m}$ $(m=1,2,\cdots;$ $k_1<k_2<\cdots)$ be as in the proof of Lemma 2. Put

$$M_\ell = \left\{ N^{h_m} v_{k_m} \;\middle|\; \begin{array}{l} m=1,2,\cdots, \\ 0\le h_m \le d-i_m \end{array} \right\} = \{ w_{ij} \text{ in } ▨ \}.$$

<u>Lemma</u> 3. <u>For</u> $\ell \in L_{\bar{k}}(\lambda)$, <u>we</u> <u>have</u>

$$S_\ell^N = \left\{ \left\langle N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mi} v_i) \;\middle|\; \begin{array}{l} m=1,2,\cdots, \\ 0\le h_m \le d-i_m \end{array} \right\rangle \;\middle|\; \begin{array}{l} a_{mi} \in K, \\ a_{mi}=0 \text{ if } v_i \in M_\ell \end{array} \right\}.$$

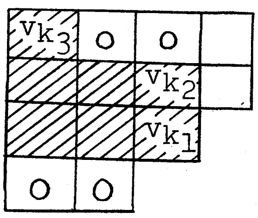
Proof. It is obvious that S_ℓ^N contains the right-hand side. Apply Lemma 1 (2) to elements of S_ℓ. Then the proof of this lemma is similar to that of Lemma 2. Thus the lemma.

<u>Definition</u> 2. Let $\ell \in L_{\bar{k}}(\lambda)$. For $v_{k_m} = w_{i_m j_m}$ $(m=1,2,\cdots)$, let $n(\ell)_m$ be the number of □ in ℓ which lies in the left-hand side of the column on which v_{k_m} lies, or in the upper position than that of $\boxed{v_{k_m}}$ in the column on which v_{k_m} lies. Put

$$n(\ell) = n(\ell)_1 + n(\ell)_2 + \cdots.$$

We remark that $n(\ell) \le e(\ell)$, where $e(\ell)$ is defined in Lemma 1 (3).

<u>Example</u>.

For $\ell =$ 

, $n(\ell)_2$ is the number of $\boxed{o}$ in ℓ.

In this case $n(\ell)_2 = n(\ell)_1 = 4$, $n(\ell)_3 = 0$ and

$$n(\ell) = n(\ell)_1 + n(\ell)_2 + n(\ell)_3 = 4 + 4 + 0 = 8.$$

In view of Lemma 1, (3), we have :

Corollary. For $\ell \in L_{\bar{k}}(\lambda)$, we have

$$S_\ell^N \simeq \mathbb{A}^{n(\ell)}.$$

Put $T_\ell = S_\ell^N$. By Lemma 3, T_ℓ is a closed subset (linear subvariety) of S_ℓ. Summing up the above statements, we have :

Theorem 1. Let the notations be as above. We have

$$G_k(V)^N = \coprod_{\ell \in L_{\bar{k}}(\lambda)} T_\ell,$$

where T_ℓ is a locally closed subset of $G_k(V)^N$ and isomorphic to an $n(\ell)$-dimensional affine space $\mathbb{A}^{n(\ell)}$.

§3. The flag manifold

Let $\mu = (\mu_1, \cdots, \mu_p, \mu_{p+1})$ be a partition of n. Put $k_j = \mu_1 + \cdots + \mu_j$ $(j = 1, 2, \cdots, p, p+1)$. Then $1 \le k_1 < k_2 < \cdots < k_p < k_{p+1} = n$. For $j = 1, 2, \cdots, p$, we denote by $\mathfrak{F}_j$ the flag manifold of type $(k_1, \cdots, k_j)$ defined by

$$\{(W_1, \cdots, W_j) \in G_{k_1}(V) \times \cdots \times G_{k_j}(V) \mid W_i \subset W_{i+1} \ (1 \le i \le j-1)\}.$$

Then, $\mathfrak{F}_j$ is isomorphic to $GL_{k_{j+1}}/P(\mu_1, \cdots, \mu_{j+1})$, where $P(\mu_1, \cdots, \mu_{j+1})$ is a parabolic subgroup of $GL_{k_{j+1}}$ whose Levi subgroup is isomorphic to $\prod_{i=1}^{j+1} GL_{\mu_i}$. In particular, if $j = p$, then $\mathfrak{F}_p \simeq GL_n/P_\mu$. For a nilpotent transformation N of V, put

$$\mathcal{F}_j^N = \{(W_i) \in \mathcal{F}_j \mid N(W_i) \subset W_i \quad (1 \le i \le j)\}.$$

If $u_\lambda = 1_n + N$ is the corresponding unipotent element of GL_n, then $\mathcal{F}_p^N \simeq (GL_n/P_\mu)_{u_\lambda}$.

We preserve the notations in §2. For $\ell \in L_{\bar{k}_p}(\lambda)$ $(\bar{k}_p = (k_p, \mu_{p+1}))$, put $V_\ell = \langle w_{ij}$ in ▨$\rangle$. We remark that V_ℓ is a element of $T_\ell = S_\ell^N$. If $W \in T_\ell$, then the projection $f : V \longrightarrow V_\ell$ induces an N-module isomorphism $f_W : W \xrightarrow{\sim} V_\ell$. By the projection

$$\pi_p : \mathcal{F}_p \longrightarrow G_{k_p}(V) \quad ((W_1, \cdots, W_p) \longmapsto W_p),$$

we have the following trivialization:

$$\pi_p^{-1}(T_\ell) \xrightarrow{\sim} \mathcal{F}_{p-1} \times T_\ell \quad ((W_i) \longmapsto (f_{W_p}(W_1), \cdots, f_{W_p}(W_{p-1})),\ W_p)).$$

Under this trivialization, we have

$$\pi_p^{-1}(T_\ell) \cap \mathcal{F}_p^N \xrightarrow{\sim} \mathcal{F}_{p-1}^N \times T_\ell,$$

and therefore $\mathcal{F}_p^N \simeq \coprod_{\ell \in L_{\bar{k}_p}(\lambda)} \mathcal{F}_{p-1}^N \times T_\ell$. By induction, we have

$$\mathcal{F}_p^N \xrightarrow{\sim} \coprod_{\substack{\ell_j \in L_{\bar{k}_j}(\lambda_j) \\ j=1,\cdots,p}} T_{\ell_1} \times T_{\ell_2} \times \cdots \times T_{\ell_p},$$

where λ_j is the Young tableau obtained by extracting the nodes with figure $j+2, \cdots, p+1$. Therefore, we can write

$$\mathcal{F}_p^N = \coprod_{\ell \in L_\mu(\lambda)} T_\ell,$$

where $L_\mu(\lambda)$ is the set of all semi-standard μ-tableaus of type λ and T_ℓ is isomorphic to some $T_{\ell_1} \times \cdots \times T_{\ell_p}$ $(\ell_j \in L_{\bar{k}_j}(\lambda_j),\ j=1,\cdots,p)$.

<u>Remark</u>. Similarly, we can prove that

$$\mathfrak{F}_p = \coprod_{\ell \in \widetilde{L_\mu(\lambda)}} S_\ell ,$$

where $\widetilde{L_\mu(\lambda)}$ is the set of all μ-tableaus of type λ. About this decomposition, we note that $T_\ell = S_\ell^N$ and $S_\ell^N \neq \phi$ if and only if $\ell \in L_\mu(\lambda)$.

<u>Definition</u> 3. For $\ell \in L_\mu(\lambda)$, let $n(\ell)$ be a non-negative integer defined by the following recurrence rule :

(1) If $\mu = (\mu_1, \mu_2)$ or (n), then $n(\ell)$ is defined in Definition 2.

(2) For $\mu = (\mu_1, \cdots, \mu_p, \mu_{p+1})$, put $\mu' = (\mu_1, \cdots, \mu_p)$ and $k_p = \mu_1 + \cdots + \mu_p$. Let $\ell_1 \in L_{\overline{k}_p}(\lambda)$ $(\overline{k}_p = (k_p, \mu_{p+1}))$ be the semi-standard $\overline{k}_p$-tableau obtained from ℓ by changing the figures $p+1$ into 2 (or □) and figures i $(1 \leq i \leq p)$ into 1 (or ▨). Let ℓ_2 be the μ'-tableau obtained by extracting the nodes with figure $p+1$ from ℓ and by rearranging the rows in the appropriate order. Thus $\ell_2 \in L_{\mu'}(\lambda_{p-1})$ for some partition λ_{p-1} of k_p. Then we defines

$$n(\ell) = n(\ell_1) + n(\ell_2).$$

<u>Theorem</u> 2. <u>Let</u> λ <u>and</u> μ <u>be a partition of</u> n. <u>The variety</u> $(GL_n/P_\mu)u_\lambda$ <u>has a partition</u>

$$(GL_n/P_\mu)u_\lambda = \coprod_{\ell \in L_\mu(\lambda)} T_\ell ,$$

<u>where</u> T_ℓ <u>is a locally closed subset of</u> $(GL_n/P_\mu)u_\lambda$ <u>and isomorphic to an</u> $n(\ell)$-<u>dimensional affine space</u> $\mathbb{A}^{n(\ell)}$ <u>and this</u>

partition is defined over K.

References

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[3] N. Spaltenstein, The fixed point set of a unipotent transformation on the flag manifold. Proc. Kon. Ak, v. Wet. 79(5), 452-456 (1976).