Difference Analogue of Volterra's Equation

Ryogo HIROTA and Masaaki ITO

Department of Applied Mathematics, Faculty of Engineering
Hiroshima University, Hiroshima

In the classic works of Volterra and Lotka the following coupled nonlinear differential equation

\[
\begin{align*}
\frac{dx}{dt} &= (\alpha - y)x, \\
\frac{dy}{dt} &= -(\beta - x)y, \\
\end{align*}
\]

is presented to describe the growth of populations of two species, prey \( x \) and predator \( y \), where \( \alpha \) and \( \beta \) are positive parameters.

The differential mapping (1) is known to exhibit the following invariant curve

\[ x + y - \beta \log x - \alpha \log y = \text{const.} \]

We look for a difference analogue of eq.(1) which exhibits an invariant curve. For this purpose we transform eq.(1) into the bilinear form and construct a difference analogue of the bilinear form using the dependent variable transformation.\(^2,3\)

Let \( x(t) = g(t)/f(t) \) and \( y(t) = h(t)/f(t) \), then eq.(1) is transformed into the following bilinear form

\[
\begin{align*}
D_t g(t) \cdot f(t) &= \alpha g(t)f(t) - g(t)h(t), \\
D_t h(t) \cdot f(t) &= -\beta h(t)f(t) + g(t)h(t), \\
\end{align*}
\]
where the bilinear operator $D^n_t$ operating on $a \cdot b$ is defined, for an integer $n$, by

$$D^n_t a(t) \cdot b(t) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^na(t)b(t') \bigg|_{t=t'}.$$ 

A difference analogue of eq.(2) is obtained by replacing the bilinear operators in eq.(2) by the difference analogues of them, namely $D_{\delta_t}$ by $\delta^{-1}[\exp(\delta D_{\delta_t}/2) - \exp(-\delta D_{\delta_t}/2)]$ and $1$ (unit operator) by $(1 - \varepsilon_1)$ exp$(\delta D_{\delta_t}/2)$ + $\varepsilon_1$ exp$(-\delta D_{\delta_t}/2)$, where $\delta$ is the time difference and $\varepsilon_1$ is a parameter, and the difference operator $\exp(\delta D_{\delta_t}/2)$ operating on $a(t) \cdot b(t)$ is defined by

$$\exp(\delta D_{\delta_t}/2)a(t) \cdot b(t) = a(t + \delta/2)b(t - \delta/2).$$ 

By these replacements, eq.(2) becomes

$$\delta^{-1}[g(t+\delta/2)f(t-\delta/2) - g(t-\delta/2)f(t+\delta/2)]$$

$$= a[(1-\varepsilon_1)g(t+\delta/2)f(t-\delta/2) + \varepsilon_1g(t-\delta/2)f(t+\delta/2)]$$

$$- [(1-\varepsilon_2)g(t+\delta/2)h(t-\delta/2) + \varepsilon_2g(t-\delta/2)h(t+\delta/2)],$$

$$\delta^{-1}[h(t+\delta/2)f(t-\delta/2) - h(t-\delta/2)f(t+\delta/2)]$$

$$= - \beta[(1-\varepsilon_3)h(t+\delta/2)f(t-\delta/2) + \varepsilon_3h(t-\delta/2)f(t+\delta/2)]$$

$$+ [(1-\varepsilon_2)g(t+\delta/2)h(t-\delta/2) + \varepsilon_2g(t-\delta/2)h(t+\delta/2)].$$

Dividing the above equations by $f(t+\delta/2)f(t-\delta/2)$, we obtain

$$\delta^{-1}[x(t+\delta/2) - x(t-\delta/2)] = a[(1-\varepsilon_1)x(t+\delta/2) + \varepsilon_1x(t-\delta/2)]$$

$$- [(1-\varepsilon_2)x(t+\delta/2)y(t-\delta/2)$$

$$+ \varepsilon_2x(t-\delta/2)y(t+\delta/2)],$$

$$\{3\}$$
\[
\delta^{-1} [y(t+\delta/2) - y(t-\delta/2)] = -\beta[(1-\varepsilon_3)y(t+\delta/2) + \varepsilon_3 y(t-\delta/2)] \\
+ [(1-\varepsilon_2)x(t+\delta/2)y(t-\delta/2) \\
+ \varepsilon_2 x(t-\delta/2)y(t+\delta/2)] ,
\]

Equation (3) is a candidate of difference analogue of Volterra's equation. Now we impose the physical condition on eq.(3) that for arbitrary value of positive \(\delta\), the populations \(x(t)\) and \(y(t)\) should be non-negative for all time if they were positive at a time. We shall select parameters \(\varepsilon_1, \varepsilon_2\), and \(\varepsilon_3\) to satisfy the condition. For small values of \(x\) and \(y\), eq.(3) is approximated by the linear equations,

\[
x(t + \delta/2) = \frac{1 + \delta \alpha \varepsilon_1}{1 - \delta \alpha (1- \varepsilon_1)} x(t - \delta/2) , \\
y(t + \delta/2) = \frac{1 - \delta \beta \varepsilon_2}{1 + \delta \beta (1- \varepsilon_3)} y(t - \delta/2) ,
\]

which show that \(x(t + \delta/2)\) and \(y(t + \delta/2)\) become negative for large values of \(\delta\) unless \(\varepsilon_1 = 1\) and \(\varepsilon_3 = 0\).

Hereafter we put \(x(t + \delta/2) = x_{t+1}\), \(x(t - \delta/2) = x_t\), \(y(t + \delta/2) = y_{t+1}\), \(y(t - \delta/2) = y_t\), \(\varepsilon_1 = 1\), \(\varepsilon_2 = \varepsilon\) and \(\varepsilon_3 = 0\), and rewrite eq.(3) as

\[
x_{t+1} - x_t = \delta[\alpha x_t - (1-\varepsilon)x_{t+1}y_t - \varepsilon x_t y_{t+1}] \\
y_{t+1} - y_t = \delta[-\beta y_{t+1} + (1-\varepsilon)x_{t+1}y_t + \varepsilon x_t y_{t+1}] .
\]

Equation (4) can be transformed into an explicit scheme for \(x_{t+1}\) and \(y_{t+1}\)

\[
x_{t+1} = \frac{[1 - \delta \varepsilon (1+\delta \beta)^{-1} x_t](1+\delta \alpha) - \delta \varepsilon (1+\delta \beta)^{-1} y_t}{1 - \delta \varepsilon (1+\delta \beta)^{-1} x_t + \delta (1-\varepsilon) y_t} x_t ,
\]

3
\[ y_{t+1} = \frac{1 + \delta(1-\varepsilon)x_{t+1}}{1 + \delta \beta - \delta \varepsilon x_t} y_t \]  

Equation (5) shows that \( x_{t+1} \) becomes negative when \( x_t \) and \( y_t \) satisfy the following conditions

\[ 1 - \delta \varepsilon(1+\delta \beta)^{-1} x_t < 0 , \]
\[ 1 - \delta \varepsilon(1+\delta \beta)^{-1} x_t + \delta(1-\varepsilon)y_t > 0 , \]

or

\[ 1 - \delta \varepsilon(1+\delta \beta)^{-1} x_t > 0 , \]
\[ (1 - \delta \varepsilon(1+\delta \beta)^{-1} x_t)(1+\delta \alpha) - \delta \varepsilon(1+\delta \beta)^{-1} y_t < 0 , \]

for positive values of \( x_t \) and \( y_t \). Hence \( \varepsilon \) must be zero. Accordingly we have a difference analogue of Volterra's equation

\[ x_{t+1} - x_t = \delta(ax_t - x_{t+1}y_t) \]
\[ y_{t+1} - y_t = \delta(-\beta y_{t+1} + x_{t+1}x_t) , \]

which reduces to eq.(1) in the small limit of \( \delta \).

Several numerical experiments carried on eq.(9) show, within experimental errors, \( (\sim 10^{-\delta}) \), that there exist invariant curves of the mapping eq.(9). We plot a typical example of them in Fig. 1.

References

