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Undecidable extensions of monadic first-order successor arithmetic (abstract)

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It is proved by R.M. Robinson [8] that the monadic second-order successor arithmetic SA[2x, x+1] with a function 2x is undecidable. Putnam proved in [6] that the monadic first-order arithmetic $FA^{=}$[P ; x+y] with the equality symbol, a single monadic predicate symbol P and a binary function + is undecidable. In contrast with these undecidability results, both the monadic second-order successor arithmetic SA[x+1] (Büchi [1]) and the monadic second-order arithmetic SA[2x] with a function 2x are decidable. The latter is an easy corollary of Rabin's result concerning the decidability of the monadic second-order theory of two successor functions [7]. So it is very interesting to search for a boundary between these decidability and undecidability results. In [5], we have shown the following ' critical ' result.

THEOREM 1. The monadic first-order successor arithmetic FA[P ; 2x, x+1] with a single monadic predicate symbol P and a function 2x (but without the equality symbol) is undecidable.

Then, the above two undecidability results follow immediately from this theorem. We will make a rough sketch of the proof of Theorem 1. Our idea is based on a reduction of the satisfiability problem of FA[P ; 2x, x+1] to the meeting problem of finite causal ω^2-systems introduced by the second author, which is shown to be unsolvable in [3]. Here, the meeting problem means the problem of deciding whether some cells will take a given special state eventually, when starting from the initial state of a given finite causal ω^2-system. For a given finite causal ω^2-system S and a given special state q_δ of S, we can construct such a formula $B_{S,\delta}$ (of FA[P ; 2x, x+1]) that $B_{S,\delta}$ is satisfiable in N (the set of natural numbers) if and only if no cells take the state q_δ in S. Thus, the unsolvability of the meeting problem implies that of the satisfiability problem of FA[P ; 2x, x+1].

A question now arises as to the generalization of Theorem 1. Can we extend this ' critical ' result, namely Theorem 1, to another monadic first-order successor arithmetic with a unary function on N ? In the following, we will give an answer to this question. The following theorem can be obtained by using the same technique as in [5].

THEOREM 2. The monadic first-order successor arithmetic $FA^=[\{P_i\}_i ; f(x), x+1]$ with the equality, countable predicate symbols $\{P_i\}_i$ and a function f(x) on N is undecidable, if the function f(x) satisfies either of the following conditions I and II.

I. 1) f is strictly monotone increasing, i.e,

$x < y$ implies $f(x) < f(y)$,

2) for some integer $r \geqq 1$, $f(x) + r < f(x + r)$ for every x.

II. 1) f is monotone increasing, i.e,

$x < y$ implies $f(x) \leqq f(y)$,

2) for some integer $r \geqq 2$, $f(x + r) < f(x) + r$ for every x,

3) for every y, there exist at least one x such that $f(x) = y$.

For example, any one of the functions x^n ($n > 1$), $[e^x]$, $[\sqrt{x}]$, $[\log x]$, $[(q/p)x]$ ($p \neq q$ and $q \neq 0$) satisfies either of the above conditions. Moreover, it turns out that many undecidability results on monadic second-order successor arithmetics, for instance, the result on a monadic second-order successor arithmetic with a *hypermonotonic* function by Elgot and Rabin [2] and some results by Siefkes [9] and Thomas [10], follow as corollaries of our undecidability results on monadic *first-order* arithmetics. We notice here the limitation of our result. The condition I.2) is equivalent to the following condition;

$$1 + \frac{1}{r} \leqq \frac{f(x + r) - f(x)}{r} .$$

Thus, if f satisfies the condition that $f'(x) > 1$ but $\lim_{x\to\infty} f'(x) = 1$ (in an approximate sense), then the condition I.2) can not be satisfied. So, for example, we don't know whether or not $FA^{=}[\ \{P_i\}_i\ ;\ x + [\sqrt{x}],\ x+1\ ]$ is undecidable. Compare this with the undecidability of $SA[\ x + [\sqrt{x}],\ x+1\ ]$ proved in [10].

The following theorem can be considered as a generalization of the decidability of SA[2x].

THEOREM 3. The monadic second-order arithmetic SA[f(x)] with a single function f(x) is decidable, if $x < f(x)$ for any x and f is strictly monotone increasing.

Remark that the above theorem holds even if f is non-recursive. We notice also that if f is strictly monotone increasing and is not identical with x, then there exists the smallest x such that $x < f(x)$. When we have an effective procedure of finding such an x, we can delete the above condition that $x < f(x)$ for any x.

Now, we will give an outline of the proof of Theorem 3. A pair $< A, f >$ is called an *algebra* (after Rabin [7]) if A is an non-empty set and f is a function defined on A. Two algebras $< A, f >$ and $< B, g >$ are *isomorphic* if there exists a bijective mapping φ from A to B such that $\varphi(f(x)) = g(\varphi(x))$ for any $x \in A$. Let F be any formula of SA[f(x)]. Then, define F* to be a formula of SA[g(x)] obtained from F by replacing every occurrence of f in F by g. Now, we have the following lemmas.

LEMMA 1. If two algebras $< N, f >$ and $< N, g >$ are isomorphic, then for any formula F of SA[f(x)], F is valid if and only if F* is valid.

LEMMA 2. If f is a strictly monotone increasing function such that $f(0) = 0$, $x < f(x)$ for any $x > 0$ and $f(x) - x$ is unbounded, then $< N, 2x >$ and $< N, f >$ are isomorphic.

By slightly modifying Lemma 2 and putting together Lemmas 1 and 2, we have Theorem 3.

Roughly speaking, it follows from Theorems 2 and 3 that we have a ' critical ' result on almost all functions which increase more *rapidly* than x. On the other hand, we can not prove an analogous result on functions which increase more *slowly* than x. Indeed, the following theorem shows a difference between them.

THEOREM 4. The first-order arithmetic $FA^{=}[\ f(x)\cdot]$ with a function f(x), and hence the monadic second-order arithmetic SA[f(x)], are undecidable for uncountably many functions f(x) satisfying the condition II of Theorem 2.

REFERENCES

[1] J.R. Büchi, On a decision method in restricted second order arithmetic, Proc. Internat. Congr. Logic, Method. and Philos. Sci. 1960, Stanford Univ. Press, 1962, 1-11.

[2] C.C. Elgot and M.O. Rabin, Decidability and undecidability of second (first) order theories of (generalized) successor, J. Symbolic Logic 31 (1966) 169-181.

[3] A. Nakamura, On causal ω^2-systems, J. of Computer and System Sciences 10 (1975) 253-265.

[4] A. Nakamura and H. Ono, Two-dimensional finite automata and their application to the decision problem of monadic first-order arithmetic A[P,f(x),g(x)], Studies on Polyautomata, (1977) 51-71.

[5] H. Ono and A. Nakamura, Undecidability of the first-order arithmetic A[P(x),2x,x+1], to appear in J. of Computer and System Sciences.

[6] H. Putnam, Decidability and essential undecidability, J. Symbolic Logic 22 (1957) 39-54.

[7] M.O. Rabin, Decidability of second-order theories and automata on infinite trees, Trans. Amer. Math. Soc. 141 (1969) 1-35.

[8] R.M. Robinson, Restricted set-theoretical definitions in arithmetic, Proc. Amer. Math. Soc. 9 (1958) 238-242.

[9] D. Siefkes, Undecidable extensions of monadic second order successor arithmetic, Zeitschr. f. Math. Logik u. Grundl. d. Math. 17 (1971) 385-394.

[10] W. Thomas, A note on undecidable extensions of monadic second order successor arithmetic, Arch. math. Logik 17 (1975) 43-44.