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The Spectrum of the Laplacian and Smooth Deformation of the Riemannian Metric

by Masao TANIKAWA

Section 1.

Let M be an n-dimensional compact connected C^∞ manifold (with or without boundary ∂M). Every Riemannian metric g of M determines a Laplace-Beltrami operator Δ_g. We consider the eigenvalue problem for Δ_g (under Dirichlet condition);

$$(1.1)\qquad \begin{cases} (-\Delta_g - \lambda)u(x) = 0 & x \in M \\ u(x) = 0 & x \in \partial M \quad (\text{ if } \partial M \neq \phi\ ) \end{cases}$$

Let $0 \leqq \lambda_0 \leqq \lambda_1 \leqq \cdots\cdots$ be the eigenvalues, which are determined by the metric g. The totality of the Riemannian metrics of class C^∞ which differ from a fixed metric g_0 only on an open set $U \subset M$ forms a separable Fréchet manifold B. Our main result is that for almost all g in B, the eigenvalues of ploblem (1.1) are all simple.

In section 2, we will discuss transversality theorem on some Fréchet manifolds. A differential operator of C^∞-type is regarded as a strong ILH mapping between Sobolev chains. We will prove theorem 2.4 which is essential for this paper.

In section 3, we will obtain the result using some techniqe originated in some similar work of K.Uhlenbeck [9], who has already obtained similar result in the case of class C^k ($n+3 \leqq k < +\infty$).

Section 2.

Let $N(d)$ be the set of all integers m satisfying $m \geq d$. We call a system $\{E, E^k, k \in N(d)\}$ a Sobolev chain, if every E^k is a Hilbert space, E^{k+1} is linerly and densely imbedded in E^k, and E is a intersection of all E^k with inverse limit topology. Let $\{E, E^k, k \in N(d)\}$, $\{F, F^k, k \in N(d)\}$ be Sobolev chains. Let U, U' be open neighbourhoods of x_o, y_o in E^d, F^d where $x_o \in E$, $y_o \in F$. Suppose a mapping $f : U \cap E \longrightarrow U' \cap F$ with $f(x_o)=y_o$.

Definition

A mapping f is called a strong ILH mapping of class C^r, if f satisfies the followings ($r \geq 2$)

(i) f can be extended to a C^r-mapping $f^k : U \cap E^k \longrightarrow U' \cap F^k$ for every $k \in N(d)$.

(ii) For any $x \in U \cap E$, there exists a E^d neighbourhood $W_x \subset U$, and for every $u \in W_x \cap E$, $v, v_1, v_2 \in E$

(2.1) $$\|(Df^k)_u v\|_k \leq C_x(\|u-x\|_k \|v\|_d + \|v\|_k) + P_x^k(\|u-x\|_{k-1})\|v\|_{k-1}$$

(2.2) $$\|(D^2f^k)_u(v_1,v_2)\|_k \leq C_x(\|u-x\|_k \|v_1\|_d \|v_2\|_d + \|v_1\|_k \|v_2\|_d + \|v_1\|_d \|v_2\|_k) + P_x^k(\|u-x\|_{k-1}) \cdot \|v_1\|_{k-1} \|v_2\|_{k-1}$$

where C_x is a positive constant and is independant of k, P_x^k is a polynomial with positive coefficients depending on k.

After the simple calculations, the composition of two C^r-strong ILH mappings is also C^r-strong ILH mapping replacing d with d+1. ILH means Inverse Limit Hilbert.

Theorem 2.1 (Implicit function theorem, H.Omori [5])

$\{E, E^k, k\in N(d)\}$, $\{F, F^k, k\in N(d)\}$, U, U', x_0, y_0, are as above. Let $f : U\cap E \longrightarrow U'\cap F$ be a C^r-strong ILH mapping with $f(x_0)=y_0$ satisfying the followings ;

(i) $(Df^k)_{x_0} : E^k \longrightarrow F^k$ is an isomorphism for every $k\in N(d)$

(ii) For every $k\in N(d)$

(2.3) $\|(Df^k)_{x_0} v\|_k \geq C\|v\|_k - D_k\|v\|_{k-1}$

where C and D_k are positive constants and C is independant of k.

Then there exist V, V' neighbourhoods of x_0, y_0 in E^d, F^d such that the mapping f is a C^r-isomorphism from $V\cap E$ into $V'\cap F$ and f^{-1} is also a C^r-strong ILH mapping satisfying inequality (2.3).

By virture of theorem 2.1, we can consider manifolds based on Sobolev chains and apply the implicit function theorem. We call such manifolds strong ILH manifolds.

A Fredholm operator is a continuous linear mapping $L : X \longrightarrow Y$ from one Banach space to another with the properties ;

(i) dim Ker L is finite

(ii) Image L is closed

(iii) Coker L = Y/Image L has finite dimension

If L is a Fredholm operator, then its index is dim Ker L - dim Coker L, so that index of L is an integer.

A C^r-strong ILH mapping $f : U\cap E \longrightarrow U'\cap F$ is a Fredholm mapping if the followings are satisfied ;

(i) For every $k \in N(d)$, every $x \in U \cap E$, $(Df^k)_x : E^k \longrightarrow F^k$ is a Fredholm operator.

(ii) The index of $(Df^k)_x$ is independant of k.

Since the set of all Fredholm operators is open in the space of all bounded operators in the norm topology and if U is connected, then the index of f is defined to be the index of $(Df^k)_x$ for some k and x.

Lemma 2.2 The notations being as above. The condition (ii) of the C^r-strong ILH Fredholm mapping can be replaced with that Ker $(Df^d)_x$=Ker $(Df^k)_x$ for every $k \in N(d)$, every $x \in U \cap E$.

proof) Ker $(Df^d)_x \supset$ Ker $(Df^k)_x$ is trivial. Since E^k, F^k are densely imbedded to E^d, F^d, codim $(Df^k)_x E^k$ = codim $(Df^d)_x E^d$. index $(Df^k)_x$ = index $(Df^d)_x$ means that dim Ker $(Df^k)_x$ = dim Ker $(Df^d)_x$, so that Ker $(Df^k)_x$ = Ker $(Df^d)_x$.

A subset of X is called residual if it is a countable intersection of open dense subsets of X. In a metric space X, a subset containing a residual set is also residual. In a complete metric space X, a residual set is dense with the Baire's theorem.

Let $f : X \longrightarrow Y$ be a C^1-mapping between two manifolds. $x \in X$ is a regular point of f if the Fréchet derivative $(Df)_x : T_x X \longrightarrow T_{f(x)} Y$ is onto, if not, x is a critical point. $y \in Y$ is a regular value if every point $x \in f^{-1}(y)$ is a regular point, if not, y is a critical value.

Theorem 2.3 (Sard's theorem)

Let U be an open set of R^p and $f : U \longrightarrow R^q$ be a C^s-mapping where $s > \max(p-q, 0)$. Then the set of critical values in R^q has measure zero.

For a proof see [6] or [8].

Our main theorem is

Theorem 2.4

Let $\{E, E^k, k \in N(d)\}$, $\{F, F^k, k \in N(d)\}$ be separable Sobolev chains, let U, U' be open sets of E^d, F^d, let $f : U \cap E \longrightarrow U' \cap F$ be a C^r-strong ILH Fredholm mapping with $r > \max(\text{index of } f, 1)$

(2.4) $\quad \|(Df^k)_x v\|_k \geq C_x \|v\|_k - D_x^k \|v\|_{k-1}$

for every $x \in U \cap E$, $v \in E$, $k \in N(d)$, where C_x, D_x^k are positive constants and C_x is independant of k.

Then the regular values of f form a residual set in F.

The proof will be given in the several lemmas below.

Lemma 2.5 Notations and assumptions being as above. f is a locally proper mapping. Namely there exists a neighbourhood W_x of E^d for every $x \in U \cap E$ such that $f^{-1}(I) \cap W_x \cap E$ is compact for any compact subset $I \subset F$.

proof) Let $x_o \in U \cap E$ be fixed, we set $A = (Df^k)_{x_o}\big|_E : E \longrightarrow F$. Since $(Df^k)_{x_o}\big|_E$ is independant of k, A is well defined and is strong ILH linear Fredholm mapping. Since dim Ker A is finite, E can be written in the form $E_1 \times \text{Ker } A$, $\{E_1, E_1{}^k, k \in N(d)\}$ is a Sobolev chain and $x_o = (p_o, q_o)$, $p_o \in E_1$, $q_o \in \text{Ker } A$. Then the first partial derivative $(D_1 f)_x : E_1 \longrightarrow F$ maps E_1 injectively onto a closed finite codimensional subspace of F for all $x = (p, q)$ sufficiently close to $x_o = (p_o, q_o)$.

Since f satisfies the inequality (2.3), we can apply the theorem 2.1 and choose a product neighbourhood $D_1 \times D_2$ of (p_o, q_o) in $E_1 \times \mathrm{Ker}\, A$ such that D_2 is compact and if $q \in D_2$ f restricted to $D_1 \times \{q\}$ is a C^r-strong ILH isomorphism onto its image. We set $W_{x_o} = D_1 \times D_2$ and $f(x_i) = y_i \longrightarrow y$, $x_i = (p_i, q_i)$ in $D_1 \times D_2$, where $i = 1, 2, 3 \cdots\cdots$. It is sufficient to show that x_i have a convergent subsequence. Since D_2 is compact we may assume $q_i \longrightarrow q$ and $f(p_i, q) \longrightarrow y$, even that $q_i = q$. But f restricted to $D_1 \times \{q\}$ is an isomorphism onto its image, so $p_i \longrightarrow p$, proving lemma 2.5.

Lemma 2.6 Let f' be the mapping f restricted to W_{x_o}. Then the regular values of f' form an open dense subset in F.

proof) In metric spaces, a proper mapping maps a closed subset to a closed subset. Since the set of all critical points of f' is closed, the regular values of f' form an open set in F. It is sufficient to show that we can find a regular value of f' in any neighbourhood V of f(x) in F. Let $\pi : F \longrightarrow F/\mathrm{Image}\, A$ be a projection and it is a strong ILH linear mapping to R^1. From the hypotheses of theorem 2.4 we can apply Sard's theorem to the mapping $\phi : p \times \mathrm{Ker}\, A \longrightarrow F/\mathrm{Image}\, A = R^1$ defined by $\phi(q) = \pi \circ f(p, q)$ to as a regular value z of ϕ in πV. Let $y \in \pi^{-1}(z) \cap V$, then y is our desired regular value.

proof of the theorem 2.4

Since E and F are separable and residual is closed under countable intersection, it is sufficient to prove locally. But lemma 2.6 has been proved.

Theorem 2.7

Let H, B and E be strong ILH manifolds of class C^r. H and B are separable. Let $f: H \times B \longrightarrow E$ be a C^r-strong ILH mapping satisfying the followings;

(i) For every $u=(h,b) \in H \times B$, every $k \in N(d)$

(2.5) $$\left\| (Df^k)_u(\delta h, \delta b) \right\|_k \geq C_u \left\| (\delta h, \delta b) \right\|_k - D_u^k \left\| (\delta h, \delta b) \right\|_{k-1}$$

where $\delta h \in T_h H$, $\delta b \in T_b B$, C_u and D_u^k are positive constants and C_u is independant of k.

(ii) There exists $e \in E$ such that e is a regular value of f.

(iii) For every $b \in B$, $f_b = f(\ ,b): H \longrightarrow E$ is a strong ILH Fredholm mapping with index of $f_b < r$.

Then the set $\{ b \in B;\ e \text{ is a regular value of } f_b \}$ is residual in B.

proof) We set $Q = f^{-1}(e) \subset H \times B$. From theorem 2.1 Q is a closed strong ILH submanifold of class C^r in $H \times B$. Let P be the projection from $Q \subset H \times B$ to B, $P(q)=b$ where $q=(h,b)$. Let i be the inclusion from Q to $H \times B$ and p be the projection from $H \times B$ to B. Note that $P = p \circ i$.

Lemma 2.8 P is a C^r-strong ILH Fredholm mapping with index of P = index of f_b.

Assuming lemma 2.8, we can apply theorem 2.4 to P which satisfies inequalities (2.4), $r >$ index of P. Then the set of regular values of P is residual in B.

$b \in B$ is a regular value of P if and only if $T_q Q = \mathrm{Ker}(Df)_q$ contains $T_b B$ for every $h \in H$ such that $(h,b)=q \in Q$. So that $(Df)_q T_h H = (Df)_q (T_h H \times T_b B) = T_e E$, namely $f_b : H \longrightarrow E$ has e as a regular value.

proof of lemma 2.8) It isn't difficult to show that P is a C^r- strong ILH mapping. We fix $k \in N(d)$ and $q=(h,b) \in Q^k$.

$T_q Q^k = \{ (\delta h, \delta b) \in T_h H^k \times T_b B^k,\ (Df^k)_q (\delta h, \delta b) = 0$

$\mathrm{Ker}\ (DP^k)_q = \mathrm{Ker}\ (Df^k)_q \cap T_h H^k = \mathrm{Ker}\ (Df_b{}^k)_q$

$\mathrm{Image}\ (DP^k)_q = \mathrm{Ker}\ (Df^k)_q \cap T_b B^k$

It is clear that $\mathrm{Ker}\ (DP^k)_q$ and $\mathrm{Image}\ (DP^k)_q$ are closed subspaces of $T_h H^k$ and $T_b B^k$. We set l=index of $f_b{}^k$, $m^k = \dim \mathrm{Ker}(Df_b{}^k)$ $n^k = \mathrm{codim}\ \mathrm{Image}(Df_b{}^k)_q$. $l = m^k - n^k$ is independant of k.

Since e is a regular value of f, there exist $b_1, b_2, \cdots\cdots b_{n^k}$ which are linealy independant such that $(Df^k)_q (0, b_j)$ $j=1,2 \cdots n^k$, and $(Df^k)_q T_h H^k$ span $T_e E$. Then $\mathrm{Ker}(Df^k)_q \cap T_b B^k$ and b_j $j=1, n^k$, span $T_b B^k$. $\mathrm{codim}\ \mathrm{Image}(DP^k)_q = n^k$ and index $P^k = m^k - n^k = l$, so that P is a Fredholm mapping with index = l.

Section 3

Let M be a compact connected n-dimensional manifold of C^{∞} class (with or without boundary ∂M). Let $C^k(M)$ denote the R-algebras of C^k-functions of M into R, and $H^k(M)$ denote the R-algebras of k-th order Sobolev functions of M into R. By the Sobolev's imbedding theorem, if $k > l + \frac{n}{2}$ and $l \geq 0$ then $H^k(M) \subset C^l(M)$ and inclusion is continuous. Let $C_o^{\infty}(M)$ denote the R-algebla of C^{∞}-functions with compact support in the interior of M. $H_o^k(M) \subset H^k(M)$ is the closure of $C_o^{\infty}(M)$.

We set $E^k = H^k(M) \cap H_o^1(M)$, $E = C^{\infty}(M) \cap H_o^1(M)$ and $H^k = H^k(M)$, $H = C^{\infty}(M)$. Then $\{ E, E^k, k \in N(d) \}$, and $\{ H, H^k, k \in N(d) \}$ are Sobolev chains for $d \geq 1$.

Let g_o be a C^{∞}-Riemannian metric on M, let dx be the volume element with respect to g_o, and let (,) be the inner product of $L^2(M,R) = H^o(M)$ by $(f,g) = \int_M f(x)g(x)dx$. Let $H^k_{\overline{U}}(M, T^*M \circledS T^*M)$ denote the totality of H^k-sections of $T^*M \circledS T^*M$ supported on $\overline{U}$, where $T^*M \circledS T^*M$ is the symmetric product of cotangent bundle T^*M, and U is an open set of M.

We fix $m > 2 + \frac{n}{2}$, choose an open neibourhood V of O in $H^m_{\overline{U}}(M, T^*M \circledS T^*M)$ such that for every $g \in g_o + V$ is a C^2-Riemannian metric on M. We set

$B^k = (g_o + V) \cap H^{k+m}(M, T^*M \circledS T^*M)$, $B = (g_o + V) \cap C^{\infty}(M, T^*M \circledS T^*M)$.

Then $g \in B^k$ is a C^{k+2}-Riemannian metric on M and $g = g_o$ on M-U. $\{ B, B^k, k \in N(d) \}$ is an open set in Sobolev chain.

Let Δ_g be the Laplace-Beltrami operator with respect to Riemannian metric g. In local coordinate with

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx_i \otimes dx_j \ , \quad g_{ij} = g_{ji} \text{ for } i,j=1,2\cdots n.$$

$$(3.1) \quad \Delta_g = \frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{G}\, g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $(g^{ij}) = (g_{ij})^{-1}$, and $G = \det(g_{ij})$.

We set $S^k = \{ u \in E^{k+2},\ (u,u)=1 \}$, $S = \{ u \in E,\ (u,u)=1 \}$. S^k is a Hilbert manifold and $\{ S,\ S^k,\ k \in N(d) \}$ is a manifold based on Sobolev chain. We consider a mapping f^k from $S^k \times R \times B^k$ to H^k given by $f^k(u,\lambda,g) = (-\Delta_g - \lambda)u$.

Lemma 3.1 If $k > \frac{n}{2}$, then f^k is well defined and is a C^∞-mapping. And for every $g \in B^k$, $f^k_g = f^k(\ ,\ ,g)$: $S^k \times R \longrightarrow H^k$ is a Fredholm mapping of index = 0.

proof) In local coordinate

$$(3.2) \quad f^k(u,\lambda,g) = -\frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{G}\, g^{ij} \frac{\partial u}{\partial x_i} \right) - \lambda u$$

If $k > \frac{n}{2}$, u is a function of class C^2 and $f^k(u,\lambda,g)$ is well defined in virtue of Sobolev's imbedding theorem. Since both G and g^{ij} are rational functions of g_{ij}, f^k is a C^∞-function of g_{ij}, u and λ.

Note that $(-\Delta_g - \lambda)$: $E^{k+2} \longrightarrow H^k$ is a Fredholm operator with index 0 for fixed λ and g. The restriction of the domain S^k gives index -1 and when λ is allowed to vary also, the final index of f^k_g is zero.

Lemma 3.2 Let f be the mapping from $S \times R \times B$ to H given by $f(u,\lambda,g) = (-\Delta_g - \lambda)u$. Then f is a strong ILH mapping of class C^∞.

proof) f^k is a natural extention of f, it is sufficient to show that f^k satisfies the inequalities (2.1) and (2.2). Using (3.2), they can be shown by a direct computation.

Lemma 3.3 f satisfies the inequality;

(3.2) $\|(Df^k)_x \delta x\|_k \geq C_g \|\delta u\|_k - D_x^k \|\delta x\|_{k-1}$

where $x=(u,\lambda,g) \in S \times R \times B$, $\delta x=(\delta u,\delta\lambda,\delta g) \in T_u S \times T_\lambda R \times T_g B$

proof) From (3.2)

$(Df^k)_x \delta x = -\Delta_g \delta u + D_2(-\Delta_g u)(\delta g) - \lambda\delta u - \delta\lambda u$

where D_2 denotes the partial derivative to g.

For the first term, using Gårding's inequality

$\|\Delta_g \delta u\|_k \geq C_g \|\delta u\|_k - D_g^k \|\delta u\|_{k-1}$

For other terms, there are no derivatives of order geater than 1, so that we obtain the inequality (3.2)

Lemma 3.4

(i) u belongs to λ-eigenspace of $-\Delta_g$, if and only if $f(u,\lambda,g)=0$.

(ii) λ-eigenspace of $-\Delta_g$ is spanned by u , if and only if $f(u,\lambda,g)=0$ and (u,λ) is a regular point of $f_g=f(\ ,\ ,g)$.

(iii) $-\Delta_g$ has only one dimensional eigenspaces if and only if $0 \in H$ is a regular value of f_g.

proof)

(i) $f(u,\lambda,g)=(-\Delta_g-\lambda)u = 0$ is a definition of λ-eigenspace of $-\Delta_g$.

(ii) u lies in a one dimensional eigenspace if and only if $(-\Delta_g-\lambda)\delta u \neq 0$ for every $\delta u \perp u$. It means that $\dim \mathrm{Ker}(Df^k_g)_{(u,\lambda)} = 0$. Since Df^k_g is a Fredholm operator with index 0, (u,λ) is a regular point of f_g.

(iii) The proof is the deductive conclusion of (i) and(ii).

Lemma 3.5 If $\dim M = n \geq 2$, then $0 \in H$ is a regular value of f.

proof) H is densely imbedded in $H^k(M)$, so that if $0 \in H$ is a regular value of f^k then the lemma 3.5 is proved. But K.Uhlenbeck has proved as to f^k [9].

Theorem 3.6

If $\dim M = n \geq 2$, then the totality of metrics $g \in B$ such that all eigenspace of $-\Delta_g$ are one dimensional is residual in B.

proof) We apply the theorem 2.7 to $f : S \times R \times B \longrightarrow H$. Lemmas 3.2, 3.3, 3.5 imply f satisfies the hypotheses of the theorem 2.7. Lemma 3.4 implies that the conclusion of the theorem 2.7 is equivalent to disired result.

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