数理解析研究所講究録
第 369 巻 1979 年 179-183

On Pretzel Links

Taizo Kanenobu
Kobe University

A link L in S^3 is said to be prime if $L = L_1 \# L_2$ implies that either L_1 or L_2 is a trivial knot. Here $L_1 \# L_2$ is a composite link of L_1 and L_2. ([N]) We will give a sufficient condition for a link to be prime and prove that pretzel links are prime.

Definition. A group G is indecomposable (relative to free products) if $G = A * B$ implies $A = 1$ or $B = 1$.

Let $\Sigma_k(L)$ be the k-fold cyclic cover of S^3 branched over a link L.

Theorem 1. If $\pi_1(\Sigma_k(L))$ is indecomposable for some $k(\geqq 2)$, then L is prime.

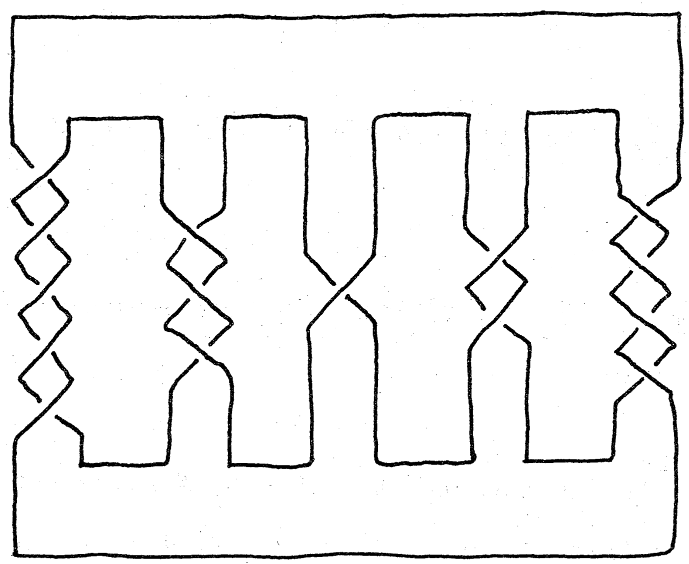
Proof. Let us suppose that $L = L_1 \# L_2$, where neither L_1 nor L_2 is a trivial knot. Then $\pi_1(\Sigma_k(L)) = \pi_1(\Sigma_k(L_1)) *$

$\pi_1(\Sigma_k(L_2))$. On the other hand, $\pi_1(\Sigma_k(L_i)) \neq 1$ for $i = 1,2$. (For a non-trivial knot, see [T]; for a link, see [HK].) This completes the proof.

<u>Corollary.</u> If $\pi_1(\Sigma_k(L))$ is a finite group or a group with a non-trivial center, in particular, an abelian group for some $k(\geqq 2)$, then L is prime.

Proof. By Problem 21 for Section 4.1 in [MKS], $\pi_1(\Sigma_k(L))$ is indecomposable.

A pretzel link $K(p_1,p_2,\ldots,p_n)$ as shown in Figure 1 is a link with a projection in which the crossings lie on n two-stranded braids, $|p_1|, |p_2|, \ldots, |p_n|$ are the numbers of crossings



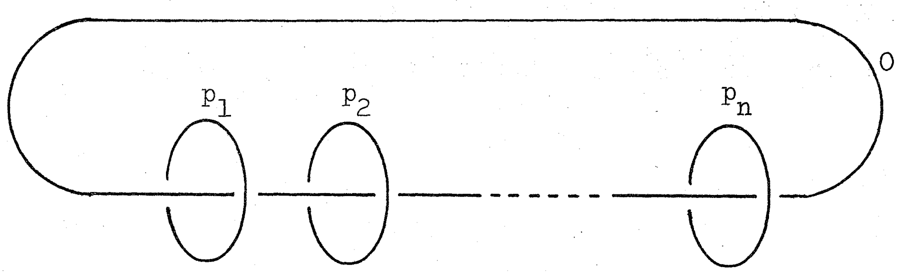
K(5,-3,1,2,-4)
Figure 1

in the braids, and the signs of $p_1, p_2, \ldots, p_n$ depend on the directions of twist in the corresponding braids.

In the projection of $K(p_1, p_2, \ldots, p_n)$, the placement of any p_i which is equal to ± 1 is immaterial insofar as link types are concerned.

A pretzel link $K(p_1, p_2, \ldots, p_n)$ is said to be degenerated if there are p_i and p_j which are equal to 1 and -1 respectively. This pretzel link is clearly equivalent to $K(p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_n)$ ($\hat{p}_i$ means that p_i is omitted).

Lemma. $\Sigma_2(K(p_1, p_2, \ldots, p_n))$ is the Seifert fiber space $(0,o,0 \mid 0;\ (p_1,1),(p_2,1),\ldots,(p_n,1))$ in Seifert's notation [S, p.208], or the manifold with the following surgery presentation:



Proof. See [Mo].

Theorem 2. A non-degenerated pretzel link $K(p_1, p_2, \ldots, p_n)$, where $n \geqq 2$ and $p_i \neq 0$ for all i, is prime.

Proof. $\pi_1(\Sigma_2(p_1, p_2, \ldots, p_n)) = \pi_1((0,o,0 \mid 0;\ (p_1,1),(p_2,1), \ldots,(p_n,1)))$ is a group with a non-trivial center or a finite group. ([O, p.92, pp.99-101])

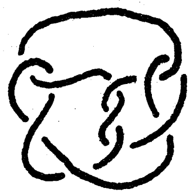
<u>Remark.</u> Most pretzel knots of type $(q_1, q_2, \ldots, q_m)$, where all the q_i and m are odd, have been shown to be prime by R. L. Parris [P].

<u>Example.</u> Let $M = \Sigma_2(10_{67})$. ([R])

Then $\pi_1(M) = \langle x, y \; ; \; y^{-1}x^4y^{-1}x^4y^{-1}x = y^3x^3y^2x^3 = 1 \rangle$. From the second relation, we have

$$x^{-3} = y^2x^3y^3 = y^3x^3y^2.$$

Thus $x^3y = yx^3$, i.e., x^3 is in the center of $\pi_1(M)$. Because $H_1(M) = \langle x \; ; \; x^{63} = 1 \rangle$, $x^3 \neq 1$ in $\pi_1(M)$, which implies that $\pi_1(M)$ has a non-trivial center. Therefore 10_{67} is prime.



10_{67}

Figure 2

References

[HK] F. Hosokawa & S. Kinoshita: On the homology of branched cyclic coverings of links, Osaka Math. J., 12 (1960), 331-355.

[MKS] W. Magnus, A. Karrass & D. Solitar: Combinatorial group theory, Interscience, New York, 1966.

[Mo] J. M. Montesinos: Variedades de Seifert que sonrecubridores ciclicos ramificados de dos hojas, Bol. Soc. Mat. Mexicana (2) 18 (1973), 1-32.

[N] Y. Nakanishi: Enumeration の話題から，本講究録．

[O] P. Orlik: Seifert Manifolds, Springer lecture notes in Math. 291.

[P] R. L. Parris: Pretzel knots, Ph. D. thesis, Princeton, 1978.

[R] D. Rolfsen: Knots and Links, Publish or Perish Inc., Berkeley, 1976.

[S] H. Seifert: Topologie dreidimensionaler gefaserter Räume, Acta Math. 60 (1933), 147-238.

[T] W. Thurston: Lectures in Conference on Smith Conjecture, Columbia Univ. (1979).