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NORMAL EULER CLASSES OF PL EMBEDDINGS WITH ISOLATED SINGULARITY

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§1. Introduction. We shall say that a PL embedding $f : V \to M$ of an n-polyhedron V into a PL (n+c)-manifold M has isolated singularity, if there is a set P of isolated points of V such that for each point x of $V - P$, there is an open neighborhood U_x of $f(x)$ in M such that $(U_x, U_x \cap f(V))$ is PL homeomorphic to the standard pair $(\mathbb{R}^{n+c}, \mathbb{R}^n \times 0)$ of euclidean (n+c)- and n-spaces. Singular set Σf of the embedding f is the minimal set P. Regular part of f is a locally flat PL embedding $f | V - \Sigma f$ of a PL n-manifold $V - \Sigma f$ into M. In this paper, we shall restrict ourselves in the case where M is oriented and V is an oriented PL n-variety, namely, $V - \Sigma f$ is connected and oriented.

A second PL embedding $f' : V \to M'$ with isolated singularity of the PL n-variety V into an oriented PL (n+c)-manifold M' is micro-isomorphic to f at a subset Q of V, if there are neighborhoods U and U' of $f(Q)$ and $f(Q')$ in M and M', respectively, and an orientation preserving PL homeomorphism $h : U \to U'$, called micro-isomorphism of f and f' at Q, such that

$$h \circ f(u) = f'(u) \quad \text{for all points } x \text{ of } f^{-1}(U).$$

In case $V = Q$, we shall say that f and f' are micro-isomorphic. The micro-isomorphism (at Q) of embeddings of V is clearly an equivalence relation.

If $\Sigma f = \emptyset$, then f is a locally flat PL embedding of a PL n-manifold V into a PL (n+c)-manifold M and the micro-isomorphism class of f is just the isomorphism class of a normal block bundle of f, which is classified by the homotopy class of its classifying map (see Rourke and Sanderson [8]).

If $\Sigma f \neq \phi$, we have obviously two invariants; the micro-isomorphism classes of f at $V - \Sigma V$ and at ΣV, respectively. The latter will be called singularity of f and denoted by $\sigma(f)$. Moreover, we may define normal euler class $X(f) \in H^c(V; \mathbf{Z})$ of f as to be a pull back of the Poincaré dual of the image of the fundamental class of V in M.

Assuming that V is a PL n-manifold (and hence c = 1 or 2 by Zeeman's unknotting theorem [11]), Noguchi [7] classified essentially the micro-isomorphism classes of such PL embeddings in terms of the singularities and the normal euler classes. As is seen from his proof in case n = c = 2, the normal euler class is linked to both of the structure of a normal block bundle of the regular part and the singularity.

It is our purpose in this paper to split the normal euler class into a certain relative euler class of the normal block bundle and (local intersection) multiplicity of the singularity.

For this, we shall prove existence and uniqueness of longitudes for locally flat PL embeddings of closed oriented connected PL manifolds into oriented spheres. This notion of longitude generalizes that of longitude for classical knots (see §2, Theorem 2.1). This enables us in §3 to define the notion of multiplicity of f at a singular point x as an invariant of the singularity of f at x, and to prove the splitting formula for the normal euler class (see Theorem 3.1). In §4, we shall extend the classification of Noguchi as follows;

Theorem. Let $f : V \to M$ be a PL embedding with isolated singularity of a compact oriented PL n-variety V into an oriented PL (n+c)-manifold M.

(1) If $c = 1$, then the singularity $\sigma(f)$ is the complete invariant of the micro-isomorphism class of f.

(2) If $c = 2$, then $\{\sigma(f), X(f)\}$ is the complete set of invariants of the micro-isomorphism classes of f.

More explicitly, the statement (2) in Theorem means that a second PL embedding $f' : V \to M'$ is micro-isomorphic to f if and only if $X(f) = X(f')$ and $\sigma(f) = \sigma(f')$, and furthermore, given a compact polyhedron V such that for a subpolyhedron P of V of $\dim P \leq 0$, $V - P$ is an oriented PL n-manifold, then for any cohomology class $\xi \in H^2(V; \mathbb{Z})$ and for any PL embedding $g : (V)_P \to \mathbb{R}^{n+2}$ with isolated singularity around P such that $\Sigma_g = P$, there is an oriented PL (n+2)-manifold M and a PL embedding $f : V \to M$ such that $\sigma(f) = \sigma(g)$ and $X(f) = \xi$.

Now let V be a compact complex variety with isolated singularity in a complex manifold M. Then a pair (M, V) admits unique oriented triangulation compatible with its complex analytic Whitney stratification, which will be denoted by the same symbol (M, V), refer to [5], p.44, Th.7). We shall say that a second pair (M', V') at $Q' \subset V'$ is micro-equivalent to (M, V) at $Q \subset V$, if there are open neighborhoods V_Q and $V'_{Q'}$ of Q and Q' in V and V', respectively, and an orientation preserving PL homeomorphism $h : V_Q \to V'_{Q'}$ such that $h(Q) = Q'$, and i and $i' \circ h$ are micro-isomorphic at Q, where $i : V \to M$ and $i' : V' \to M'$ are inclusion maps. If V is a hypersurface in M, then by Theorem, (M, V) and (M', V') are micro-equivalent if and only if there is an orientation preserving PL homeomorphism

$g : V \to V'$ such that $\sigma(i) = \sigma(i' \circ g)$ and $g^{*}X(i') = X(i)$. In general, it would be a deep problem to find a such PL homeomorphism g.

Nevertheless, in case V and V' are curves in a complex surface we have

Corollary to Theorem. Let C and C' be compact irreducible complex curves in a complex surface S.

Suppose that C and C' represent the same homology class in S. Then (S, C) and (S', C') are micro-equivalent if and only if (S, C) at ΣC and (S, C') at $\Sigma C'$ are micro-equivalent, where ΣC and $\Sigma C'$ are singular sets of C and C', respectively.

In particular, if $S = S'$ is a complex projective plane $\mathbb{P}^2$, then $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are micro-equivalent if and only if they are micro-equivalent at singular sets and have the same self-intersection number.

§2, Longitudes and multiplicity.

Let $F : M \to S^{m+c}$ be a locally flat PL embedding of an oriented closed PL m-manifold M into an oriented (m+c)-sphere. Let ν be a normal c-block bundle of F over a simplicial division K of M, and let $\dot{\nu}$ be a (c-1)-sphere block bundle associated to ν (for block bundles see [8]). We shall denote the total space of a block bundle ξ by the same symbol ξ.

Definition. For a block bundle ξ over a complex L, a PL map $\varphi : |L| \to \xi$ is a (block) section of ξ, if $\varphi(\sigma) \subset \xi|\sigma$ (the block over σ) for all $\sigma \in L$.

It is known that for the PL embedding $F : M \to S^{m+c}$, the normal sphere block bundle $\dot{\nu}$ over K admits a section over the c-skeleton $K^{(c)}$ of K, since its euler class vanishes. We would like to specify a section of $\dot{\nu}$ restricted to the (c-1)-skeleton $K^{(c-1)}$ of K which extends to a section of $\dot{\nu}$ restricted to the c-skeleton $K^{(c)}$ of K by making use of the Alexander duality in S^{m+c}.

(1) Suppose that M is connected.

Definition. A section $\varphi : K^{(c-1)} \to \dot{\nu} | K^{(c-1)}$ is a longitude of $F : M \to S^{m+c}$, if an induced chain map $\varphi_{\#} : \sum_{k=0}^{c-1} C_k(K) \to \sum_{k=0}^{c-1} S_k(\dot{\nu})$ can be extended to a chain map

$$\Phi : \sum_{k=0}^{c} C_k(0*K) \to \sum_{k=0}^{c} S_k(E),$$

where $C_k(K)$ is a simplicial k-chain complex of K, $S_k(X)$ for a space X is a singular k-chain complex of X, $E = S^{m+c} - \mathrm{Int}\,\nu$ and $0*K$ is a cone complex of K with vertex 0.

Theorem 2.1. Let $F : M \to S^{m+c}$ be a locally flat PL embedding of an oriented closed connected PL m-manifold into an oriented (m+c)-sphere. Then for any normal block bundle ν of F over a simplicial division K of M, there is a longitude $\varphi : K^{(c-1)} \to \dot{\nu} | K^{(c-1)}$ of F unique up to homotopy of sections.

Proof. We have that by the general positive reason

$$\pi_i(S^{m+c}, E) = 0 \quad \text{for all } i \leqq c-1,$$

and by the Alexander duality,

$$H_i(E, \dot{\nu} | \sigma) = 0 \quad \text{for all } i \leq c-1,$$

and

$$H_{c-1}(\dot{\nu} | \sigma) \cong H_{c-1}(E) = \mathbb{Z},$$

since M is connected. Since $\pi_i(\dot{\nu} | \sigma) \cong \pi_i(E) = 0$ for

$i \leqq c-2$, we have a section $\psi : K^{(c-1)} \to \dot{\nu} \mid K^{(c-1)}$ whose induced chain map $\psi_{\#} : \sum_{k=0}^{c-1} C_k(K) \to \sum_{k=0}^{c-1} S_k(\dot{\nu})$ can be extended to a chain map

$$\Psi : \sum_{k=0}^{c-1} C_k(0*K) \longrightarrow \sum_{k=0}^{c-1} S_k(E).$$

In order to extend $\Psi \mid \sum_{k=0}^{c-2} C_k(0*K)$ to a chain map $\sum_{k=0}^{c} C_k(0*K) \longrightarrow \sum_{K=0}^{c} S_k(E)$, we have an obstruction theory with coefficients in $H_{c-1}(E; \mathbb{Z}) = \mathbb{Z}$ over a simplicial chain complex $0*K$. Since $0*K$ is acyclic, it follows that $\Psi \mid \sum_{k=0}^{c-2} C_k(0*K)$ can be extended to a chain map

$$\Phi : \sum_{k=0}^{c} C_k(0*K) \longrightarrow \sum_{k=0}^{c} S_k(E).$$

From the fact that $H_{c-1}(E, \dot{\nu} \mid \sigma) = 0$ and $\pi_{c-1}(\dot{\nu} \mid \sigma) = H_{c-1}(\dot{\nu} \mid \sigma; \mathbb{Z}) = \mathbb{Z}$ for all $\sigma \in K$, we may assume that

$$\Phi(\sigma) \in S_{c-1}(\dot{\nu} \mid \sigma) \quad \text{for each } (c-1)\text{-simplex } \sigma \text{ of } K,$$

and $\Phi \mid \sum_{k=0}^{c-1} C_k(K)$ is induced from a section

$$\varphi : K^{(c-1)} \longrightarrow \dot{\nu} \mid K^{(c-1)},$$

which is the required longitude.

Now let $\varphi' : K^{(c-1)} \longrightarrow \dot{\nu} \mid K^{(c-1)}$ be a second longitude of F. Since $\dot{\nu} \mid \sigma$ is (c-2)-connected for each simplex σ of K, we have a homotopy $\eta : \varphi \mid K^{(c-2)} \simeq \varphi' \mid K^{(c-2)}$ of sections which induces a chain map of degree 1

$$\eta_{\#} : \sum_{k=0}^{c-2} C_k(K) \longrightarrow \sum_{k=1}^{c-1} S_k(\dot{\nu})$$

such that

$$\eta_{\#}(\sigma) \in S_{k+1}(\dot{\nu} \mid \sigma) \quad \text{and} \quad \partial \eta_{\#}(\sigma) = \eta_{\#}(\partial \sigma) + \varphi_{\#}(\sigma) - \varphi'_{\#}(\sigma)$$

for each k-simplex $(k \leqq c-2)$ σ of K. By the same reason as above, $\eta_{\#} \mid \sum_{k=0}^{c-3} C_k(K)$ can be extended to a chain map of degree 1

$$\eta : \sum_{k=0}^{c-1} C_k(0*K) \longrightarrow \sum_{k=1}^{c} S_k(E)$$

such that

$\eta(\sigma) \in S_{c-1}(\dot{\nu}|\sigma)$ for each (c-2)-simplex σ of K

and $\eta \,|\, \sum_{k=0}^{c-2} C_k(K)$ is induced from a homotopy $\xi : \varphi \,|\, K^{(c-2)} \simeq \varphi' \,|\, K^{(c-2)}$ of sections. Let $d(\varphi, \varphi')$ be an obstruction to extending the homotopy ξ to a homotopy $\varphi \simeq \varphi'$ of sections. Then we have that for each (c-1)-simplex σ of K,

$$d(\varphi,\varphi')(\sigma) = L(F_\#[M],\ \eta_\#(\partial\sigma)+\varphi_\#(\sigma)-\varphi'_\#(\sigma))$$

(linking number of $F_\#[M]$ and $\eta_\#(\partial\sigma)+\varphi_\#(\sigma)-\varphi'_\#(\sigma)$ in S^{m+c})

$$= L(F_\#[M], \partial\eta(\sigma)) = 0,$$

since $\eta(\sigma) \in S_c(E)$, where $[M]$ is the fundamental class of M. Hence φ and φ' are homotopic as sections, completing the proof.

Remark 1. By the last arguments in the proof above, a longitude $\varphi : K^{(c-1)} \longrightarrow \dot{\nu} \,|\, K^{(c-1)}$ can be extended to a section $\tilde{\varphi} : K^{(c)} \longrightarrow \dot{\nu}|K^{(c)}$.

(2) Suppose that M is not connected.

Let $M_1, \ldots, M_r$ be the connected components of M and let ν_i be a normal block bundle of $F_i = F \,|\, M_i$ over a simplicial division K_i of M_i. For each $i = 1, \ldots, r$, we have a longitude $\varphi_i : K_i^{(c-1)} \longrightarrow \dot{\nu} \,|\, K_i^{(c-1)}$ of $F_i : M_i \longrightarrow S^{m+c}$. We put $E_i = S^{m+c} - \mathrm{Int}\, \nu_i$ and E_{ij} ($= E_{ji}$) $= E_i - \mathrm{Int}\, \nu_j$. Note that $H_k(E_{ij}) = 0$ for $k \leqq c-2$ and $H_{c-1}(E_{ij}) \cong H_{c-1}(E_i) \oplus H_{c-1}(E_j) \cong \mathbf{Z} \oplus \mathbf{Z}$. For each i, j, we have an obstrcution $m_j(\varphi_i) \in C^c(0*K_i, K_i; H_{c-1}(E_{ij}))$ to extending $(\varphi_i)_\#$ to a chain map $\sum_{k=0}^{c} C_k(0*K_i) \longrightarrow \sum_{k=0}^{c} S_k(E_{ij})$. Since φ_i is a longitude of F_i

and unique up to homotopy of sections, its cohomology class, denoted by the same symbol $m_j(\varphi_i)$, is well-defined as an element of $H^c(0*K_i, K_i; H_{c-1}(E_j)) = H^c(0*K_i, K_i; \mathbb{Z})$. Thus we have a cohomology class

$$m(\varphi_i) = \sum_{j=1}^{r} m_j(\varphi_i) \in H^c(0*M_i, M_i; \mathbb{Z})$$

and a cohomology class

$$m(F) = \sum_{i=1}^{r} m(\varphi_i) \in H^c(0*M, M; \mathbb{Z}).$$

Let $f_0 = 0*F : 0*M \to 0'*S^{m+c}$ be a cone extension of F. Regarding of $m(F)$ as an element of $H^c(0*M, 0*M-0; \mathbb{Z})$ (by deformation retraction $M \simeq 0*M-0$), we shall call the class $m(F)$ as to be (local intersection) <u>multiplicity</u> of f_0 at 0 and denot it by $m(f_0)$. The following will be proved by the standard argumen and the proof will be left to the reader;

<u>Proposition</u> 2.2. (i) The multiplicity $m(f_0)$ is invariant under micro-isomorphism of f_0 at 0.

(2) If $c \geqq (m+1)+1$, then $m(f_0) = 0$.

(3) If $c = m+1$ $(m \geqq 1)$, then we have that for each local orientation $[0*M_i] \in H_{m+1}(0*M_i, 0*M_i-0; \mathbb{Z})$,

$$m(f_0)([0*M_i]) = \sum_{\substack{j=1 \\ j\neq i}}^{r} L(F_\#[M_j], (\varphi_i)_\#[M_i]) \quad (\text{in } S^{2m+1}).$$

(4) In particular, if $r = 1$, then $m(f_0) = 0$.

§3. <u>Normal euler classes and the splitting formula</u>.

Let $f : V \to M$ be a PL embedding with isolated singularit of a compact oriented PL n-variety V into an oriented PL (n+c)-manifold M. Suppose that $n \geqq 2$. The fundamental class

$[V] \in H_n(V; \mathbf{Z}) = \mathbf{Z}$ of V is determined by the orientation of $V - \Sigma V$.

<u>Definition</u>. We define euler class $X(f)$ of f as to be an integral cohomology class

$$X(f) = f^* \circ j'^* \circ P_M^{-1} \circ f_*[V] \in H^c(V; \mathbf{Z}),$$

where $P_M : H^c(M; \mathbf{Z}) \to H_n(M; \mathbf{Z})$ is the Poincaré duality isomorphism of M determined by the orientation of M, and $j' : M \to (M, \partial M)$ is an inclusion map.

Let $f : L \to K$ be a simplicial division of the PL embedding $f : V = |L| \to M = |K|$. Then a singular set Σf of f consists of vertices of L. For the second barycentric subdivisions K'', L'' of K, L and for each point x of Σf, we put

$$V_x = st(x, L''), \quad B_x = st(f(x), K''), \quad \ell_x = \ell k(x, L''),$$
$$S_x = \ell k(f(x), K''), \quad f_x = f \mid V_x : V_x \to B_x \quad \text{and}$$
$$\dot{f}_x = f \mid \ell_x : \ell_x \to S_x.$$

Note that ℓ_x is a closed PL $(n-1)$-manifold (called a <u>link</u> of x in V), B_x is a PL $(n+c)$-ball whose boundary $\partial B_x = S_x$ is a PL $(n+c-1)$-sphere and $f_x : V_x \to B_x$ is a cone extension

$$x * \dot{f}_x : x * \ell_x \to f(x) * S_x$$

of a locally flat PL embedding $\dot{f}_x : \ell_x \to S_x$. We put $V_\Sigma = \bigcup_x V_x$, $B_\Sigma = \bigcup_x B_x$, $S_\Sigma = \bigcup_x S_x$, $\ell_\Sigma = \bigcup_x \ell_x$, $f_\Sigma = \bigcup_x f_x :$ $V_\Sigma \to B_\Sigma$, $\dot{f}_\Sigma = \bigcup_x \dot{f}_x : \ell_\Sigma \to S_\Sigma$, $V_0 = (V - V_\Sigma) \cup \ell_\Sigma$, and $M_0 = (M - B_\Sigma) \cup S_\Sigma$, where x ranges over all points of ΣV.

In the following, we shall denote subcomplexes of L'' covering $V_0, V_\Sigma, \ell_\Sigma, \ldots$ by the same symbols $V_0, V_\Sigma, \ell_\Sigma, \ldots,$ respectively.

Following the construction of normal block bundles ([8] or [3], Part I), we have a decomposition of a regular neighborhood $N = st(f(V), K'')$ of $f(V)$ in M; $N = \nu \cup B_\Sigma$ such that ν is a normal block bundle of a locally flat PL embedding $f_0 = f \mid V_0 : V_0 \to M_0$ over V_0 and $\nu \cap B_\Sigma$ is a restriction of ν over ℓ_Σ, denoted by ν_Σ, which is a normal block bundle of $\dot{f}_\Sigma : \ell_\Sigma \to S_\Sigma$ over ℓ_Σ. Considering of B_x as to be a block over V_x, we shall refer N together with the decomposition $\nu \cup B_\Sigma$ as to be a (block) stratified regular neighborhood of $f : V \to M$, compare with Stone [10].

For each component $\ell_{x,1}, \dots, \ell_{x,r_x}$ of ℓ_x $(x \in \Sigma f)$ we take a longitude $\varphi_{x,i} : \ell_{x,i}^{(c-1)} \to \dot{\nu} | \ell_{x,i}^{(c-1)}$ and put $\varphi_x = \bigcup_{i=1}^{r_x} \varphi_{x,i}$ and $\varphi_\Sigma = \bigcup_{x \in \Sigma f} \varphi_x : \ell_\Sigma^{(c-1)} \to \dot{\nu} | \ell_\Sigma^{(c-1)}$.

Definition. We define relative euler class $X(\nu, \varphi_\Sigma) \in H^c(V_0, \partial V_0; \mathbb{Z})$ of (ν, φ_Σ) as to be an obstruction to extending the section φ_Σ to a section $V_0^{(c)} \to \dot{\nu} | V_0^{(c)}$.

On the other hand, we have multiplicity

$$m(f_x) \in H^c(V_x, V_x - x; \mathbb{Z})$$

of $f | V_x$ at $x \in \Sigma f$. We regard of $X(\nu, \varphi_\Sigma)$ as an element of $H^c(V; \mathbb{Z})$ $(\cong H^c(V, \Sigma V; \mathbb{Z}) \cong H^c(V, V_\Sigma; \mathbb{Z}) \cong H^c(V_0, \partial V_0; \mathbb{Z}))$. Let $k_x^* : H^c(V_x, V_x - x) \cong H^c(V, V-x) \to H^c(V)$ be a natural homomorphism.

Theorem 3.1 (The splitting formula for $X(f)$).

Suppose that $n \geqq 2$. Then we have that

$$X(f) = X(\nu, \varphi_\Sigma) + \sum_{x \in \Sigma f} k_x^* m(f_x).$$

Proof. We have a section $\Phi_0 : V_0^{(c-1)} \to \dot{\nu} | V_0^{(c-1)}$ extending $\varphi_\Sigma : \ell_\Sigma^{(c-1)} \to \dot{\nu} | \ell_\Sigma^{(c-1)}$. Since $\nu | \sigma$ is contractible for each simplex σ of V_0, Φ_0 can be extended to a section $\Psi_0 : V_0 \to \nu$ such that for each c-simplex σ of V_0, $\Psi_0(\sigma)$ intersects transversally to $f(V_0)$ in ν. On the other hand, since S_x is (n+c-2)-connected for each point $x \in \Sigma V$, we can extend $\Psi_0 | \ell_\Sigma : \ell_\Sigma \to \nu | \ell_\Sigma$ to a PL map $\Psi_\Sigma : V_\Sigma \to S_\Sigma$ such that for each c-simplex σ of $V_\Sigma - \ell_\Sigma$, $\Psi_\Sigma(\sigma)$ intersects $f(\ell_\Sigma)$ transversally. Moreover, chain maps

$$f_\# : \sum_{k=0}^{n} C_k(V) \to \sum_{k=0}^{n} S_k(\nu \cup B_\Sigma)$$

and

$$(\Psi)_\# : \sum_{k=0}^{n} C_k(V) \to \sum_{k=0}^{n} S_k(\nu \cup B_\Sigma)$$

are chain homotopic, where $\Psi = \Psi_0 \cup \Psi_\Sigma$.

Let $U \in H^c(N, \partial N; \mathbb{Z})$ be the Poincaré dual of $f_*[V]$ in N. It is not hard to see that

$$X(f) = f^* \circ (j)^* \circ P_N^{-1} \circ f_*[V] = f^* \circ (j)^*(U) = \Psi^* \circ (j)^*(U),$$

where $j : N \to (N, \partial N)$ is an inclusion map.

Let σ be a c-simplex of V.

If $\sigma \in V_0$, we have that

$$X(f)(\sigma) = \Psi^\# \circ (j)^\#(U)(\sigma) = (j)^\#(U)(\Psi_\# \sigma)$$

= intersection number of $f(V_0)$ and $\Psi_\# \sigma$ in ν

= $X(\nu, \varphi_\Sigma)(\sigma)$, and

if $\sigma \in V_\Sigma - \ell_\Sigma$, we have that if $\sigma \in V_x - \ell_x$ $(x \in \Sigma f)$,

$$X(f)(\sigma) = (j)^\#(U)(\Psi_\# \sigma)$$

= intersection number of $f_\#[V_x]$ and $\Psi_\#(\sigma)$ in B_x

= intersection number of $f_\#[\ell_x]$ and $\Psi_\#(\sigma)$ in S_x

= $\sum_{i=1}^{r_x}$ linking number of $f_*[\ell_{x,i}]$ and $\partial \Psi_\#(\sigma)$ in S_x

= $m(\varphi_x)(\sigma) = m(f_x)(\sigma)$.

This proves that $X(f) = X(\nu, \varphi_\Sigma) + \sum_{x \in \Sigma V} k_x^* \, m(f_x)$.

§4. Proofs of Theorem and Corollary.

First of all, we would like to give a general method to get a micro-equivalence.

Hypothesis 1. Suppose that $\sigma(f) = \sigma(f')$. Then we have simplicial divisions

$f : L \to K$ and $f' : L \to K'$ of $f : V \to M$ and $f' : V \to M'$,

respectively, such that there is an isomorphism $h_\Sigma : B_\Sigma \to B'_\Sigma$ of f_Σ and f'_Σ; $h_\Sigma \circ f_\Sigma = f'_\Sigma$. By the uniqueness of normal block bundles, we may assume that $h_\Sigma(\nu_\Sigma) = \nu'_\Sigma$, and $h_\Sigma | \nu_\Sigma : \nu_\Sigma \to \nu'_\Sigma$ is a block isomorphism.

Note that if we can choose h_Σ so that $h_\Sigma | \nu_\Sigma : \nu_\Sigma \to \nu'_\Sigma$ extends to a block isomorphism $h_0 : \nu \to \nu'$, then we have the required micro-isomorphism $h : N \to N'$ of f and f' by setting $h | \nu = h_0$ and $h | B_\Sigma = h_\Sigma$. In order to describe the obstruction to doing this, we make

Hypothesis 2. Suppose that ν_Σ is trivial and there is a block isomorphism $h_0 : \nu \to \nu'$. Then we have a block isomorphism

$$g = h_0^{-1} \circ h_\Sigma : \nu_\Sigma \to \nu'_\Sigma,$$

which can be identified with a semi-simplicial map $\gamma : \ell_\Sigma \to S\widetilde{PL}_c$ from $\partial V_0 = \ell_\Sigma$ to the structural group $S\widetilde{PL}_c$ of oriented c-block bundles.

Notice that γ is null homotopic if and only if g can be extended to a block isomorphism

$G : \nu \to \nu$ such that $G(u) = u$ for each point u of ν restricted to the outside of a collar neighborhood of $\partial V_0 = \ell_\Sigma$ in V_0.

Hypothesis 3. The map γ is null homotopic.

Then we have a block isomorphism $G : \nu \longrightarrow \nu$ as above and $h_0 \circ G : \nu \longrightarrow \nu'$ is an extension of $h_\Sigma | \nu_\Sigma : \nu_\Sigma \rightarrow \nu'_\Sigma$.

Proof of Theorem.

In case $c = 1$, the structural group $S\widetilde{PL}_1$ is obviously of the homotopy type of one point. It follows that $\sigma(f) = \sigma(f')$ implies that f and f' are micro-equivalent.

In case $c = 2$, the structural group $S\widetilde{PL}_2$ of oriented 2-block bundles has the homotopy type of a circle $S^1 = K(\mathbf{Z}, 1)$ (refer to [8] or, partially, [3], Part II) and hence the classifying space $BS\widetilde{PL}_2$ is $K(\mathbf{Z}, 2)$. Thus for a polyhedron Y, a homotopy set $[Y, BS\widetilde{PL}_2]$ (= the set of all isomorphism classes of 2-block bundles over Y) can be identified with a cohomology group $H^2(Y; \mathbf{Z})$ by $\xi \mapsto X(\xi)$ = the euler class of ξ (the primary obstruction to constructing a section of ξ over a 2-skeleton of Y). As for ν_Σ , we have that $X(\nu_\Sigma) = 0$ and hence ν_Σ is trivial, because of the existence of a longitude. Moreover, a (total) longitude $\varphi_\Sigma : \ell_\Sigma^{(c-1)} \rightarrow \dot{\nu} | \ell_\Sigma^{(c-1)}$ can be taken as to be a PL embedding which extends to a trivialization $\Phi : \varepsilon^2(\ell_\Sigma) \rightarrow \nu_\Sigma$, where $\varepsilon^2(\ell_\Sigma)$ is a product PL disk bundle over ℓ_Σ.

We have a 2-block bundle

$$(\nu , \Phi) = \nu \underset{\Phi}{\cup} \varepsilon^2(V_\Sigma)$$

from a disjoint union of ν and $\varepsilon^2(V_\Sigma)$ by identifying $\varepsilon^2(\ell_\Sigma)$ with ν_Σ via the isomorphism Φ .

The euler class $X(\nu, \Phi)$ of (ν, Φ) coincides with the relative euler class $X(\nu, \varphi_\Sigma)$. The assumption $\sigma(f) = \sigma(f')$ implies that $k_x^* m(f_x) = k_x'^* m(f_x')$ for each $x \in \Sigma f$. Thus $X(f) = X(f')$ implies that by the splitting formula, $X(\nu, \Phi) = X(\nu', \Phi')$. Since V_x is contractible for each $x \in \Sigma f$, there is an isomorphism $h_0 : \nu \longrightarrow \nu'$ such that

$$h_0 \circ \Phi | \varepsilon^2(\ell_\Sigma) = \Phi' | \varepsilon^2(\ell_\Sigma)$$

if and only if $X(\nu, \Phi) = X(\nu', \Phi')$.

On the other hand, from the uniqueness of longitudes

$$h_\Sigma \circ \varphi_\Sigma : \ell_\Sigma^{(1)} \longrightarrow \dot{\nu}' | \ell_\Sigma^{(1)}$$

is again a longitude of $\dot{f}'_\Sigma : \ell_\Sigma \longrightarrow S'_\Sigma$ which is homotopic to φ'_Σ as sections. Since $\widetilde{SPL}_2$ is $K(\mathbb{Z}, 1)$, this implies that

$$\Phi'^{-1} \circ h_\Sigma \circ \Phi | \varepsilon^2(\ell_\Sigma)$$

represents a trivial element of $(H^1(\ell_\Sigma ; \mathbb{Z}) =) [\ell_\Sigma , \widetilde{SPL}_2]$. Thus $h_0^{-1} \circ h_\Sigma$ (identified with $\Phi'^{-1} \circ h_0^{-1} \circ h_\Sigma \circ \Phi$) represents the trivial element of $[\ell_\Sigma, \widetilde{SPL}_2]$. It follows that by the arguments above we have a micro-equivalence $h : N \longrightarrow N'$ of f and f'.

Suppose that we are given V, P, ξ and $g : (V)_P \longrightarrow \mathbb{R}^{n+2}$ as in the explanation of Theorem. The micro-equivalence class of g at the singular set $\Sigma g = P$ is represented by

$$g_\Sigma : V_\Sigma \longrightarrow B_\Sigma .$$

Let $\Phi : \varepsilon^2(\ell_\Sigma) \longrightarrow \nu_\Sigma$ be a trivialization of a normal block bundle ν_Σ of $\dot{g}_\Sigma : \ell_\Sigma \longrightarrow S_\Sigma$. On the other hand, we have an oriented disk bundle η over V such that $X(\eta) = \xi$. By the same reason as above, we have a trivialization $\Psi : \varepsilon^2(V_\Sigma) \longrightarrow \eta | V_\Sigma$.

We construct a compact oriented (n+2)-manifold M from a disjoint union of $\eta | V_0$ and B_Σ by identifying $\eta | \ell_\Sigma$ and ν_Σ via an isomorphism $\Psi \circ \Phi^{-1} | \nu_\Sigma : \nu_\Sigma \longrightarrow \eta | \ell_\Sigma$, and a PL embedding $f : V \longrightarrow M$ by setting

$$f | V_0 = \text{the zero-section of } \eta | V_0$$

and $$f | V_\Sigma = g | V_\Sigma .$$

It is clear from the construction that M and f are the required ones, completing the proof.

Proof of Corollary. We take stratified regular neighborhoods $\nu \cup B_\Sigma$, $\nu' \cup B'_\Sigma$ of C, C' in S, respectively. First of all, from the assumption that (S, C) at ΣC and (S, C') at $\Sigma C'$ are micro-equivalent, we take an orientation preserving PL homeomorphism

$$h_\Sigma : (B_\Sigma, C_\Sigma) \longrightarrow (B'_\Sigma, C'_\Sigma).$$

We put $\Sigma C = \{x_1, \ldots, x_s\}$ and $\Sigma C' = \{y_1, \ldots, y_s\}$, where $y_i = h_\Sigma(x_i)$, $i = 1, \ldots, s$. We may assume that $h_\Sigma | (B_{x_i}, C_{x_i})$ is a cone extension of $\dot{h}_{x_i} = h_\Sigma | (S_{x_i}, \ell_{x_i})$ for each x_i.

Now let $\rho : \tilde{C} \to C$ and $\rho' : \tilde{C}' \to C'$ be normalizations of C and C', respectively. By ([4], Theorem B), we have that

$$\chi(\tilde{C}) = \chi(C) + \sum_{i=1}^{s} (r_i - 1) = (i^* c^1(S) - X(i)) \cap [C] + \sum_{i=1}^{s} \mu_i + \sum_{i=1}^{s} (r_i - 1)$$

$$= c^1(S) \cap i_*[C] - \langle i_*[C], i_*[C] \rangle + \sum_{i=1}^{s} (\mu_i + r_i - 1),$$

where [χ stands for the euler number,] $c^1(S)$ is the first chern class of S, $i : C \to S$ is an inclusion map, $\langle i_*[C], i_*[C] \rangle$ is the self-intersection number of $i_*[C]$ in S, μ_i is the Milnor number at the singular point x_i and r_i is the number of connected components of a link ℓ_{x_i} of x_i in C.

Note that μ_i and r_i are invariant under micro-equivalence of (S, C) at x_i. From the assumptions that $i_*[C] = i'_*[C']$ and (S, C) at ΣC and (S, C') at $\Sigma C'$ are micro-equivalent we have that $\chi(\tilde{C}) = \chi(\tilde{C}')$. Hence there is an orientation preserving PL homeomorphism

$$\tilde{g} : \tilde{C} \to \tilde{C}' .$$

Since $\rho^{-1}(C_{x_i})$ is a disjoint union of disks $\widetilde{C}_{i,j}$, $j = 1, \ldots, r_i$, for each $i = 1, \ldots, s$, by the homogeneity of disks on a connected surface we may assume that $\widetilde{g}(\widetilde{C}_{i,j}) = \widetilde{C}'_{i,j}$ for i, j. By the isotopy theorem of PL homeomorphisms of PL balls, we may further assume that

$$\rho' \circ \widetilde{g} \mid \widetilde{C}_{i,j} = h_\Sigma \circ \rho \mid \widetilde{C}_{i,j},$$

refer to [2]. Therefore $\widetilde{g} : \widetilde{C} \longrightarrow \widetilde{C}'$ gives rise to a PL homeomorphism $h : C \longrightarrow C'$ extending h_Σ so that $\sigma(i) = \sigma(i' \circ h)$. Since $[C'] = h_*[C]$, we have that $X(i) = h^* X(i')$. Therefore, by Theorem, i and $i' \circ h$ are micro-equivalent, completing the proof.

Remark 2. The formula;

$$\chi(\widetilde{C}) = c^1(S) \cap i_*[C] - \langle i_*[C], i_*[C] \rangle + \sum_{i=1}^{s} (\mu_i + r_i - 1)$$

in the proof above is equivalent to the adjunction formula;

$$2 - 2g(C) = -(K + C) \cdot C + \sum_{i=1}^{s} 2\delta_i$$

in the theory of complex curves in complex surfaces by passing to the Milnor-Jung formula;

$$2\delta_i = \mu_i + r_i - 1,$$

refer to Serre ([9], Lemma 2, p.74) and Milnor ([6], Theorem 10.5), where K and C are the canonical line bundle of S and the line bundle over S determined by a divisor C, and $g(C)$ is the genus of $C \equiv$ the genus of $\widetilde{C}$.

Remark 3. Let V be a complex analytic subset with isolated singularity of a complex manifold M. For a point x of V, suppose that $\dim_{\mathbb{C}} V_x = n$ and $\dim_{\mathbb{C}} M = n+c$.

According to Barth [1], if $n - c - 1 \geq 0$, then a link ℓ_x

of x in V is connected. Therefore, the multiplicity of V in M at x vanishes, provided that $n \neq c$.

Remark 4. Let $f : V \to M$ be a PL embedding with isolated singularity of an n-polyhedron into an oriented PL (n+c)-manifold. Suppose that $V - \Sigma f$ is oriented but not connected. Then we have the irreducible components $V_1, \ldots, V_r$ of V as the closures of all the connected components of $V - \Sigma f$. Putting $f_i = f | V_i$, we have that $\Sigma f_i = \Sigma f \cap V_i$. In case $c \leq 2$, since the complete set of invariants of each f_i is induced from the complete set of invariants of f by the restriction, it follows that Theorem still holds in case $V - \Sigma f$ is not connected. However, Corollary should be modified as follows;

Corollary*. Let C and C' be complex curves in a complex surface S with irreducible decompositions $C = C_1 \cup \cdots \cup C_r$ and $C' = C_1' \cup \cdots \cup C_r'$. Suppose that C_i and C_i' represent the same homology class in S for each $i = 1, \ldots, r$. Then, there is a micro-equivalence of (S, C) and (S, C') inducing micro-equivalences (S, C_i) and (S, C_i'), $i = 1, \ldots, r$, if and only if there is a micro-equivalence of (S, C) at ΣC and (S, C') at $\Sigma C'$ inducing micro-equivalences of (S, C_i) at ΣC_i and (S, C_i') at $\Sigma C_i'$ respectively.

Remark 5. Let V be a complex hypersurface in a complex projective (n+1)-space $\mathbb{P}^{n+1}$. If $\Sigma V = \emptyset$, then the diffeomorphism class of an oriented pair $(\mathbb{P}^{n+1}, V)$ is completely determined by the homology class of V in $\mathbb{P}^{n+1}$ (or the degree of V).

In ([12], Chap. III, §3, p.214 and §5), Zariski gives an example of a curve V ($n=1$) such that the micro-equivalence class of $(\mathbb{P}^{n+1}, V)$

$\neq$ the PL homeomorphism class of $(\mathbb{P}^{n+1}, V)$.

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