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Deformations of $\mathbb{C}^*$-Seifert fibrations

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We describe deformations of $\mathbb{C}^*$-Seifert fibrations and make a remark on the relationship between deformations of an isolated singularity (X,p) with $\mathbb{C}^*$ action and deformations of the $\mathbb{C}^*$-Seifert fiber space X-p. For the torus Seifert fibering case, we refer to [8]. Details of this note will appear elsewhere.

§1. $\mathbb{C}^*$-Seifert fibrations.

Following Conner-Raymond [2], we construct $\mathbb{C}^*$-Seifert fiberings as follows. Let W be a complex manifold and let N be a group acting analytically and properly discontinuously on W from the left. The quotient space V=N\W has a natural structure of complex space and the projection $\nu: W \to V$ is holomorphic. We assume that V is compact hereafter. Consider the exponential sequence

$$(1.1) \qquad 0 \to Z \xrightarrow{1} \mathbb{C} \xrightarrow{\varepsilon} \mathbb{C}^* \to 1,$$

$(\varepsilon(z)=\exp 2\pi iz)$. Contrary to the torus case, we let N act trivially on each of the groups in (1.1). By taking the sheaf of germs of holomorphic maps from W into each of the groups

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in (1.1), we get the exact sequence of sheaves over W

(1.2) $$0 \to \mathcal{Z} \to \mathcal{O}_W \to \mathcal{O}_W^* \to 1.$$

If $\mathcal{S}$ is one of the sheaves in (1.2), we define, for each $\alpha \in N$ and an open set U in W,

$$\alpha : \Gamma(U,\mathcal{S}) \to \Gamma(\alpha U,\mathcal{S})$$

by $(\alpha\sigma)(w)=\sigma(\alpha^{-1}w)$, $\sigma \in \Gamma(U,\mathcal{S})$, $w \in \alpha U$. Then we have a structure of N-sheaf on $\mathcal{S}$ ([2],[3],[8]). From (1.2), we get the cohomology exact sequence

(1.3) $$\cdots \xrightarrow{\iota_*} H^1(N,\mathcal{O}_W) \xrightarrow{\varepsilon_*} H^1(N,\mathcal{O}_W^*) \xrightarrow{c} H^2(N,\mathcal{Z}) \longrightarrow \cdots.$$

We call Ker c the Picard group of the action (N,W) and denote it by Pic(N,W). Each element m in the group $H^1(N,\mathcal{O}_W^*)$ defines a principal $\mathbb{C}^*$-bundle $\varpi:B \to W$ and an action (N,B) covering (N,W) as follows. First, if we take a suitable open covering $\{W_\lambda\}_{\lambda\in\Lambda}$ of W, m is represented by a collection $\{m^{\lambda\mu}(w;\alpha)\}$, where for each $(\lambda,\mu)\in\Lambda^2$ and $\alpha\in N$, $m^{\lambda\mu}(\ ;\alpha)$ is a non-vanishing holomorphic function on $W_\lambda \cap W_\mu$. The collection satisfies the cocycle condition

(1.4) $$m^{\lambda\nu}(w;\alpha\beta) = m^{\lambda\mu}(w;\alpha)m^{\mu\nu}(\alpha^{-1}w;\beta),$$

$(\lambda,\mu,\nu)\in\Lambda^3$, $(\alpha,\beta)\in N^2$. In particular, if we set $\alpha=\beta=e$ (the identity of N), we get

$$m^{\lambda\nu}(w;e) = m^{\lambda\mu}(w;e)m^{\mu\nu}(w;e).$$

Thus the collection $\{m^{\lambda\mu}(w;e)\}$ defines a principal $\mathbb{C}^*$-bundle $\varpi:B \to W$ with $\varpi^{-1}(W_\lambda) \simeq W_\lambda\times\mathbb{C}^*$. We let N act on $W_\lambda\times\mathbb{C}^*$ by

(1.5) $$\alpha(w,t^\lambda) = (\alpha w,m^{\lambda\lambda}(\alpha w;\alpha)t^\lambda),$$

$\alpha \in N$, $(w,t^{\lambda}) \in W_{\lambda} \times \mathbb{C}^*$. Then we get an action of N on $\tilde{\varpi}^{-1}(W_{\lambda})$. The cocycle condition (1.4) shows that the actions $(N,\tilde{\varpi}^{-1}(W_{\lambda}))$ and $(N,\tilde{\varpi}^{-1}(W_{\mu}))$ coincide on $\tilde{\varpi}^{-1}(W_{\lambda} \cap W_{\mu})$ and we get a global action (N,B) covering (N,W). Clearly the action is properly discontinuous. It is fixed point free if and only if the isotropy subgroup N_w has no fixed points on the fiber $\tilde{\varpi}^{-1}(w)(\simeq \mathbb{C}^*)$, i.e. if c(m) is a Bieberbach class ([2]). Thus if c(m) is a Bieberbach class, the quotient $M=N\backslash B$ is a complex manifold. Since the action (N,B) is compatible with the canonical right action of $\mathbb{C}^*$ on B, M admits a $\mathbb{C}^*$-action and we have the diagram

$$\begin{array}{ccc} (N,B,\mathbb{C}^*) & \xrightarrow{\mu} & (M,\mathbb{C}^*) = (N\backslash B,\mathbb{C}^*) \\ \tilde{\varpi}\downarrow & & \downarrow\pi \\ (N,W) & \xrightarrow{\nu} & V = N\backslash W = M/\mathbb{C}^*. \end{array} \tag{1.6}$$

We call $M \xrightarrow{\pi} V$ the $\mathbb{C}^*$-Seifert fibration determined by m. The fiber $\pi^{-1}(\nu(w))$ over a point $\nu(w) \in V$ is given by $\pi^{-1}(\nu(w))=N_w\backslash\tilde{\varpi}^{-1}(w) \simeq N_w\backslash\mathbb{C}^*$. When $N_w \neq \{e\}$, we call the fiber a (multiple) singular fiber of $M \to V$. It is not difficult to show

<u>Lemma 1. 1.</u> Given a principal $\mathbb{C}^*$-bundle $\tilde{\varpi}:B \to W$ and properly discontinuous actions of N on B and W so that $\tilde{\varpi}$ is equivariant. Assume that the action (N,B) is compatible with the canonical $\mathbb{C}^*$ action. Then there is an element m in $H^1(N, \mathcal{O}^*_W)$ such that (N,B) is equivalent to the one constructed from m as above.

<u>§2. Deformations of $\mathbb{C}^*$-Seifert fibrations.</u>

<u>Definition 2. 1.</u> Let

$$\begin{array}{ccc} B & \longrightarrow & M \\ \downarrow & & \downarrow \\ W & \longrightarrow & V \end{array}$$

be a $\mathbb{C}^*$-Seifert fibration as constructed in §1. A deformation of it consists of

(I) A deformation $\mathcal{B} \xrightarrow{\Pi} \mathcal{W} \xrightarrow{\omega} S$ of the principal $\mathbb{C}^*$-bundle $\varpi : B \to W$ ([4] Definition 1.8) (we let $o \in S$ be the specific point so that $\omega^{-1}(o) = W_o \simeq W$, $\Pi^{-1}(W_o) = B_o \simeq B$ and $\Pi | B_o \simeq \varpi$),

(II) Properly discontinuous actions $(N, \mathcal{B})$ and $(N, \mathcal{W})$ such that

(a) Π and ω are equivariant (we let N act trivially on S),

(b) $(N, B_o) \simeq (N, B)$ and $(N, W_o) \simeq (N, W)$,

(c) $(N, \mathcal{B})$ is compatible with the canonical $\mathbb{C}^*$ action on $\mathcal{B}$.

If we set $W_s = \omega^{-1}(s)$ and $B_s = \Pi^{-1}(W_s)$ for each $s \in S$, Lemma 1.1 shows that (N, B_s) is equivalent to the one constructed from a cohomology class $m(s) \in H^1(N, \mathcal{O}^*_{W_s})$. Since $m(s)$ depends "holomorphically" on s, if $c(m)$ is a Bieberbach class so is $c(m(s))$ for every sufficiently small s. Therefore if S is "small", the quotient $\mathcal{M} = N \backslash \mathcal{B}$ is a complex manifold. We have the diagram

$$\begin{array}{ccc} N \backslash \mathcal{B} = \mathcal{M} & & \\ \downarrow & \searrow & \\ & & N \backslash \mathcal{W} = \mathcal{V} \\ S & \longleftarrow & \end{array}$$

where $\mathcal{M} \longrightarrow \mathcal{V}$ is a deformation of $M \to V$ over S. Given a deformation as in Definition 2.1, then we have, from (I), the Kodaira-Spencer fundamental sheaf diagram ([4] $(4.2)_P$)

$$
(2.1)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Theta_W & \longrightarrow & \Pi_W & \longrightarrow & T_W & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma_W & \longrightarrow & \Gamma_W & \longrightarrow & T_W & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Xi_W & \longrightarrow & \Xi_W & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

If we denote by $\mathbb{T}_X$ the holomorphic tangent bundle of a complex manifold, we have $\Theta_W = \mathcal{O}_W(\mathbb{T}_W)$, $\Sigma_W = \mathcal{O}_W(\mathbb{T}_B/\mathbb{C}^*)$ and $T_W = W \times \mathbb{T}_{S,o}$ ($\mathbb{T}_{S,o}$=the holomorphic tangent space of S at o). Moreover, if we denote by $\mathbb{T}_{B/W}$, the bundle of tangent vectors of B which are tangential to the fibers of ϖ, we have $\Xi_W = \mathcal{O}_W(\mathbb{T}_{B/W}/\mathbb{C}^*)$. Note that each sheaf in (2.1) has a natural structure of N-sheaf. From the second row we get the connecting homomorphism $\delta : H^0(N, T_W) \to H^1(N, \Sigma_W)$. Since N acts trivially on S, we have $H^0(N, T_W) = \mathbb{T}^N_{S,o} = \mathbb{T}_{S,o}$. Thus we get the infinitesimal deformation map

$$\eta : \mathbb{T}_{S,o} \longrightarrow H^1(N, \Sigma_W).$$

It is not difficult show that $H^1(N, \Sigma_W)$ is the set of isomorphism classes of first order infinitesimal deformations of the $\mathbb{C}^*$-Seifert fibration $M \to V$. Noting that $\Xi_W \simeq \mathcal{O}_W$, we have, from the first column of (2.1), the cohomology exact sequence

$$\cdots \longrightarrow H^0(N, \Sigma_W) \xrightarrow{\psi^0} H^0(N, \Theta_W) \xrightarrow{\delta^0} H^1(N, \mathcal{O}_W)$$

$$\xrightarrow{\phi^1} H^1(N, \Sigma_W) \xrightarrow{\psi^1} H^1(N, \Theta_W) \xrightarrow{\delta^1} H^2(N, \mathcal{O}_W) \longrightarrow \cdots.$$

If we set $C = \mathrm{Ker}\ \psi^1 = \mathrm{Im}\ \phi^1$, $F = \mathrm{Ker}\ \delta^1 = \mathrm{Im}\ \psi^1$, we get a decomposition of $H^1(N,\Sigma_W)$ into vector groups

(2.2) $$0 \longrightarrow C \longrightarrow H^1(N,\Sigma_W) \longrightarrow F \longrightarrow 0.$$

As in [8] §4, we can show that C represents the Picard deformations of $M \to V$ and that none of elements in C is obstructed. The group $H^1(N,\Theta_W)$ is the set of isomorphism classes of first order infinitesimal deformations of the action (N,W) and the map δ^1 gives the first obstruction to constructing a deformation of $M \to V$ from the given deformation of (N,W). Thus we may say that F represents the "base deformations" of $M \to V$.

§3. $\dim_{\mathbb{C}} W = 1$.

When W is one dimensional, the groups C and F are computed as follows. We may assume that W is simply connected without loss of generality. Let g denote the genus of the compact Riemann surface $V=N\backslash W$. The image of the set $\{w \in W \mid N_w \neq \{e\}\}$ by the map ν consists of a finite number of points $p_1, \cdots, p_r$ on V. Let d denote the divisor $\sum_{i=1}^{r} p_i$ on V and let $\Theta_{V|d}$ denote the sheaf of germs of holomorphic vector fields on V which vanish on d. By [8] Lemma 2.1 and Proposition 3.4, we have

$$H^p(N,\Theta_W) = H^p(V,\Theta_{V|d}) \text{ for } p \geq 0.$$

Theorem 3. 1.

$$\dim C = \begin{cases} 0 \cdots g = 1,\ r = 0 \text{ and } \psi^0 = 0 \\ g \cdots \text{otherwise.} \end{cases}$$

If $g=1$ and $r=0$, then $W=\mathbb{C}$, $N=\mathbb{Z}^2$ and V is a complex torus. We give a condition for ψ^0 to be zero in this case. First, if $W=\mathbb{C}$, then $B=W\times\mathbb{C}^*$ and $H^1(N, \mathcal{O}_W^*) \simeq H^1(N, H^0(W, \mathcal{O}_W^*))$. Therefore, the element m defining (N,B) can be also represented by a crossed homomorphism $m:N \to H^0(W, \mathcal{O}_W^*)$. Set $m(w;\alpha)=m(\alpha)(w)$ and $M(w,\alpha)=\frac{1}{2\pi i}\log m(w;\alpha)$. Then for each $\alpha\in N$, $\frac{dM}{dw}(w;\alpha)$ is a single valued holomorphic function on W. The map $\frac{dM}{dw}$ which assigns $\frac{dM}{dw}(w;\alpha)$ to each α is a crossed homomorphism from N into $H^0(W, \mathcal{O}_W)$.

Proposition 3. 2. When $g=1$ and $r=0$, ψ^0 is non-zero if and only if $\frac{dM}{dw}$ is a principal crossed homomorphism.

Using

Lemma 3. 3. $H^p(N, \mathcal{O}_W)=0$ when $p\geq 2$,

we get

Theorem 3. 4.

$$\dim F = \dim H^1(V,\Theta_{V|d}) = \begin{cases} 0 \cdots g = 0 \text{ and } r \leq 3, \\ 1 \cdots g = 1 \text{ and } r = 0, \\ 3g-3+r \cdots \text{otherwise.} \end{cases}$$

§5. Deformations of isolated singularities with $\mathbb{C}^*$ action.

Let X be an affine algebraic variety over $\mathbb{C}$ with an isolated singular point p and a $\mathbb{C}^*$ action. In this section we consider the relationship between deformations of (X,p) and deformations of the $\mathbb{C}^*$-Seifert fibration $X - p \to X - p/\mathbb{C}^*$. For simplicity, we assume that X is a homogeneous cone with vertex p. Thus for a suitable embedding $(X,p)\subset(\mathbb{C}^{n+1},0)$, the action $(\mathbb{C}^*,X)$ is given by $t(z_o,z_1,\cdots, z_n)=(tz_o,tz_1,\cdots, tz_n)$. Also we assume that p is a normal point of X and that $\dim X\geq 2$. Let I be the ideal of X and let $N_X=\mathcal{H}om_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$ be the normal sheaf of X in $\mathbb{C}^{n+1}$. The isomorphism classes of first order deformations of (X,p) is given by the Schlessinger's T^1_X, which is defined by the exact sequence ([1],[6],[7])

$$H^0(X,\Theta_{\mathbb{C}^{n+1}}|_X) \to H^0(X,N_X) \to T^1_X \to 0. \tag{4.1}$$

Setting $B=X-p$, we get the diagram

$$\begin{array}{ccc} B & \subset & \mathbb{C}^{n+1}-0 \\ \tilde{\omega}\downarrow & & \downarrow p \\ W & \subset & \mathbb{P}^n \end{array} \quad ,$$

where W is defined by I in $\mathbb{P}^n$, p is the projection of the universal $\mathbb{C}^*$-bundle and $\tilde{\omega}$ is the restriction of p to B. Note that

$$H^0(X,\Theta_X) \simeq H^0(B,\Theta_B),\ H^0(X,\Theta_{\mathbb{C}^{n+1}}|_X) \simeq H^0(B,\Theta_{\mathbb{C}^{n+1}}|_B)$$

and

$$H^0(X,N_X) \simeq H^0(B,N_B)$$

([6],[7]). If we denote by N_W the normal sheaf of W in $\mathbb{P}^n$, we get $N_B=\tilde{\omega}^*N_W$. Hence we have $H^0(B,N_B)=H^0(B,\tilde{\omega}^*N_W)=\sum_{\nu=-\infty}^{\infty} H^0(W,N_W(\nu))$. Also from $\Theta_{\mathbb{C}^{n+1}}|_B \simeq \tilde{\omega}^*\mathcal{O}_W(1)^{n+1}$, we get $H^0(B,\Theta_{\mathbb{C}^{n+1}}|_B)=\sum_{\nu=-\infty}^{\infty} H^0(W, \mathcal{O}_W(\nu+1))^{n+1}$. Thus we get a grading $T^1_X=\sum_{\nu=-\infty}^{\infty} T^1_X(\nu)$ ([5],[6],[7]). $T^1_X(0)$ is defined by the exact sequence

(4.2) $$H^0(W, \mathcal{O}_W(1))^{n+1} \to H^0(W,N_W) \to T^1_W(0) \to 0.$$

On the other hand, dividing the sheaf exact sequence

$$0 \to \Theta_B \to \Theta_{\mathbb{C}^{n+1}}\big|_B \to N_B \to 0$$

over B by $\mathbb{C}^*$, we get the sheaf exact sequence

$$0 \to \Sigma_W \to \mathcal{O}_W(1)^{n+1} \to N_W \to 0$$

over W. From this we get the exact sequence

(4.3) $$\cdots \to H^0(W, \mathcal{O}_W(1))^{n+1} \to H^0(W,N_W) \to H^1(W,\Sigma_W)$$
$$\to H^1(W, \mathcal{O}_W(1))^{n+1} \to \cdots.$$

Comparing (4.2) and (4.3), we get

$$T^1_X(0) = H^1(W,\Sigma_W),$$

if $H^1(W, \mathcal{O}_W(1))=0$. In this case, the set of first order infinitesimal deformations of $B \to W$ coincides with the set of first order infinitesimal deformations of (X,p) in which the $\mathbb{C}^*$ action on X is stable (extendible). The condition is satisfied, for example, if W is a complete intersection of dimension greater than one or if W is a plane curve of degree less than four.

The non-homogeneous case can be dealt with by taking a suitable covering of X.

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