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Openness Condition for Filtered Complexes and A Comparison Theorem

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In this note, we will prove an abstract comparison theorem concerning the completion of a filtered complex of abelian groups under a certain hypothesis of "openness" for the differentiation, and its extension to the case of an inductive system of filtered complexes. The latter is a generalization of a key lemma of N. Sasakura [3].

We use the notation and terminology of A. Grothendieck [1].

1. Statement of the results.

Let $K^{\cdot}$ be a filtered complex of abelian groups :

$$K^{\cdot} \supset \ldots \supset F^{p}K^{\cdot} \supset F^{p+1}K^{\cdot} \supset \ldots \quad (p \in \mathbb{Z}).$$

By definition, the completion $K^{\cdot\wedge}$ of $K^{\cdot}$ is the projective limit $\varprojlim_{p} K^{\cdot}/F^{p}K^{\cdot}$. As a type of completion of the cohomology $H^{i}(K)$ ($i \in \mathbb{Z}$), we take the projective limit $\varprojlim_{p} H^{i}(K/F^{p}K)$. Then, by the universal property of $\varprojlim_{p}$, there exists a canonical funtorial homomorphism

$$\psi^{i} : \quad H^{i}(K^{\wedge}) \longrightarrow \varprojlim_{p} H^{i}(K/F^{p}K)$$

for each degree $i \in \mathbb{Z}$.

We say that $K^\cdot$ satisfies the openness condition (B_i), if there exists a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$F^{f(p)}K^i \cap B^i(K) \subset d^{i-1}(F^pK^{i-1})$$

for all $p \in \mathbb{Z}$. The condition (B_i) is nothing but the openness of the differentiation $d^{i-1} : K^{i-1} \longrightarrow B^i(K)$, $B^i(K)$ being regarded as endowed with the filtration induced by that of K^i. As a weaker condition, we say that $K^\cdot$ satisfies the weak openness condition (WB_i), if there exists a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$F^{f(p)}K^i \cap B^i(K) \subset \bigcap_q (F^qK^i \cap B^i(K) + d^{i-1}(F^pK^{i-1}))$$

for all $p \in \mathbb{Z}$. In topological terms, the condition (WB_i) is equivalent to saying that $d^{i-1} : K^{i-1} \longrightarrow B^i(K)$ maps every neighborhood of the zero to a subset whose closure is a neighborhood of the zero. We will prove the following

Theorem I. Let $K^\cdot$ be a filtred complex of abelian groups. Assume that $K^\cdot$ satisfies the weak openness condition (WB_i) for a degree $i \in \mathbb{Z}$. Then the canonical homomorphism

$$\psi^i : H^i(K^\wedge) \longrightarrow \varprojlim_p H^i(K/F^pK)$$

is an isomorphism.

We remark that the (weak) openness condition plays as a substitute for the Artin-Rees theorem in the case of modules of finite type over a commutative Noetherian ring.

We will extend this comparison theorem to the case of an inductive system of filtered complexes.

Let $(K^\cdot_\alpha, u_{\beta\alpha})$ be an inductive system of filtered complexes of abelian groups indexed by a directed set ($u_{\beta\alpha} : K^\cdot_\alpha \longrightarrow K^\cdot_\beta$ for

$\alpha \leq \beta$). We say that $(K^{\cdot}_{\alpha})_{\alpha}$ satisfies the openness condition (B^{*}_{i}), if, for every index α, there exist a $\beta \geq \alpha$ and a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$u_{\beta\alpha}(F^{f(p)}K^{i}_{\alpha} \cap B^{i}(K_{\alpha})) \subset d^{i-1}(F^{p}K^{i-1}_{\beta})$$

for all $p \in \mathbb{Z}$. We say that $(K^{\cdot}_{\alpha})_{\alpha}$ satisfies the weak openness condition (WB^{*}_{i}), if, for every index α, there exist a $\beta \geq \alpha$ and a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$u_{\beta\alpha}(F^{f(p)}K^{i}_{\alpha} \cap B^{i}(K_{\alpha})) \subset \bigcap_{q} (u_{\beta\alpha}(F^{q}K^{i}_{\alpha} \cap B^{i}(K_{\alpha})) + d^{i-1}(F^{p}K^{i-1}_{\beta})),$$

for all $p \in \mathbb{Z}$. In this case, our result is

<u>Theorem II.</u> Let $(K^{\cdot}_{\alpha}, u_{\beta\alpha})$ be an inductive system of filtred complexes of abelian groups indexed by a directed set. Assume that $(K^{\cdot}_{\alpha})_{\alpha}$ satisfies the weak openness condition (WB^{*}_{i}) for a degree $i \in \mathbb{Z}$. Then the canonical homomorphism

$$\psi^{i} = \varinjlim_{\alpha} \psi^{i}_{\alpha} \; : \; H^{i}(\varinjlim_{\alpha} \hat{K_{\alpha}}) \longrightarrow \varinjlim_{\alpha} \varprojlim_{p} H^{i}(K_{\alpha}/F^{p}K_{\alpha})$$

is an isomorphism.

This Theorem II is a generalization of Prop. 2.1 in [3]

Though we will prove our theorems under the weak openness conditions (WB) and (WB^{*}), the openness conditions (B) and (B^{*}) might be more useful in applications. That is why we detailed the latter. We remark that our way of proof is valid in a more general setting of categories.

2. Proof of Theorem I.

We use the right derived functor of $\varprojlim_p$: (projective systems of abelian groups indexed by $\mathbb{Z}$) $\longrightarrow$ (abelian groups), which we denote by $R^i\varprojlim_p$. For this derived functor, we refer to R. Hartshorne [2], Chapter I, §4. Note that $R^i\varprojlim_p = 0$ for $i \geq 2$

Let $K^\cdot$ be a filtered complex of abelian groups. Applying Proposition (4.4) (loc. cit.) to the projective system $(K^\cdot/F^pK^\cdot)_p$, we get an exact sequence

$$(2.1) \qquad 0 \longrightarrow R^1\varprojlim_p H^{i-1}(K/F^pK) \longrightarrow H^i(K^\wedge) \xrightarrow{\psi^i} \varprojlim_p H^i(K/F^pK) \longrightarrow 0$$

for each degree $i \in \mathbb{Z}$. (A rapid way to derive this exact sequence is to compare the two spectral sequences which converge to the hypercohomology $\mathbb{R}^i\varprojlim_p K^\cdot/F^pK^\cdot$.) This leads us to the study of the kernel of ψ^i.

We propose to replace $R^1\varprojlim_p H^{i-1}(K/F^pK)$ by an abelian group which represents how far the differentiation $d^{i-1} : K^{i-1} \longrightarrow B^i(K)$ is from openness. First, note that the short exact sequence of complexes

$$0 \longrightarrow F^pK^\cdot \longrightarrow K^\cdot \longrightarrow K^\cdot/F^pK^\cdot \longrightarrow 0$$

induces the cohomology exact sequence

$$(2.2) \qquad \cdots \longrightarrow H^i(F^pK) \longrightarrow H^i(K) \longrightarrow H^i(K/F^pK) \longrightarrow H^{i+1}(F^pK) \longrightarrow \cdots$$

for each p. We set

$$(2.3) \qquad L^i(K)_p = \operatorname{Ker}\,(H^i(F^pK) \longrightarrow H^i(K))$$

and

$$F^pH^i(K) = \operatorname{Im}\,(H^i(F^pK) \longrightarrow H^i(K)).$$

Then, from the long exact sequence (2.2), we get an exact sequence

$$(2.4) \qquad 0 \longrightarrow H^{i-1}(K)/F^pH^{i-1}(K) \longrightarrow H^{i-1}(K/F^pK) \longrightarrow L^i(K)_p \longrightarrow 0$$

for each $i \in \mathbb{Z}$. We regard (2.4) as an exact sequence of projective systems indexed by p. Since the projective system in the second term of (2.4) consists of epimorphisms, its $R^j\varprojlim_p$ vanish for $j \geq 1$. Hence, passing to the limit, the exact sequence (2.4) assures an isomorphism

$$(2.5) \qquad R^1\varprojlim_p H^{i-1}(K/F^pK) \xrightarrow{\sim} R^1\varprojlim_p L^i(K)_p.$$

With this identification, we get an exact sequence

$$(2.6) \qquad 0 \longrightarrow R^1\varprojlim_p L^i(K)_p \longrightarrow H^i(K^\wedge) \longrightarrow \varprojlim_p H^i(K/F^pK) \longrightarrow 0,$$

in place of (2.1). Note that this exact sequence (2.6) is functorial in $K^\cdot$.

Recall that a projective system (M_p, π_{pq}) $(\pi_{pq} : M_q \longrightarrow M_p$, for $p \leq q)$ indexed by $\mathbb{Z}$ is said to satisfy the Mittag-Leffler condition (ML) if there exists a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ $(f(p) \geq p)$ such that $\operatorname{Im}\pi_{pq} = \operatorname{Im}\pi_{pf(p)}$, for all p and $q \geq f(p)$. If $(M_p)_p$ satisfies (ML), then $R^1\varprojlim_p M_p = 0$. (loc. cit.) As a more restrictive condition, we say that $(M_p)_p$ is essentially zero, if there exis ts a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that the homomorphism $\pi_{pf(p)} : M_{f(p)} \longrightarrow M_p$ is zero for all $p \in \mathbb{Z}$ (or, equivalently, if $(M_p)_p$ is isomorphic to zero as a pro-object). It is easy to verify that, if $(M_p)_p$ is essentially zero, then $\varprojlim_p M_p = o$ and $R^1\varprojlim_p M_p = 0$. (The implication

$$(2.7) \qquad (M_p)_p \text{ is essentially zero.} \Longrightarrow (M_p)_p \text{ satisfies (ML).}$$

is clear.) Note that the notion "essentially zero" is stable under functorial operations.

Returning to the filtred complex $K^\bullet$, we focus on the projective system $(L^i(K)_p)_p$. By the definition (2.3), we have

$$L^i(K)_p = \frac{F^pK^i \cap B^i(K)}{d^{i-1}(F^pK^{i-1})}. \tag{2.8}$$

Rewrite the two conditions "essentially zero" and (ML) for this $(L^i(K)_p)_p$, using the formulae (2.8). Then, we have the following dictionary :

(2.9) $K^\bullet$ satisfies (B_i). $\Longleftrightarrow$ $(L^i(K)_p)_p$ is essentially zero.

(2.10) $K^\bullet$ satisfies (WB_i). $\Longleftrightarrow$ $(L^i(K)_p)_p$ satisfies (ML).

((2.9) implies (2.10). cf. (2.7).)

Now, the Theorem I is clear. If $K^\bullet$ satisfies (WB_i), then $(L^i(K)_p)_p$ satisfies (ML) and $R^1\varprojlim_p L^i(K)_p = 0$. By the exact sequence (2.6), we have an isomorphism

$$\psi^i : H^i(K^\wedge) \xrightarrow{\sim} \varprojlim_p H^i(K/F^pK).$$

3. Proof of Theorem II.

Let $(K^\bullet_\alpha, u_{\beta\alpha})$ be an inductive system of filtered complexes of abelian groups indexed by a directed set. Then, by the exact sequence (2.6), we get an exact sequence of inductive systems indexed by α

$$0 \longrightarrow R^1\varprojlim_p L^i(K_\alpha) \longrightarrow H^i(K^\wedge_\alpha) \xrightarrow{\psi^i_\alpha} \varprojlim_p H^i(K_\alpha/F^pK_\alpha) \longrightarrow 0 \tag{3.1}$$

for each degree $i \in \mathbb{Z}$.

We modify the definitions of "essentially zero" and (ML) for this case. Let $(M_{p,\alpha} : \pi^{\alpha}_{pq}, u^{p}_{\beta\alpha})$ be an inductive system, indexed by a direcred set, of projective systems indexed by $\mathbb{Z}$. Here, p and α indicate the indices as a projective system and as an inductive system, re$\overset{s}{\wedge}$pectively.

$$\begin{array}{ccc} M_{p,\alpha} & \xrightarrow{u^{p}_{\alpha\beta}} & M_{p,\beta} \\ {\scriptstyle \pi^{\alpha}_{pq}} \uparrow & & \uparrow {\scriptstyle \pi^{\beta}_{pq}} \\ M_{q,\alpha} & \xrightarrow{u^{q}_{\alpha\beta}} & M_{q,\beta} \end{array}$$

$(p \leq q, \ \alpha \leq \beta)$

We say that the system $(M_{p,\alpha})_{p,\alpha}$ is essentially zero, if, for every index α, there exist a $\beta \geq \alpha$ and a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ ($f(p) \geq p$) such that the homomorphism $\pi^{\beta}_{pf(p)} \circ u^{f(p)}_{\beta\alpha} = u^{p}_{\beta\alpha} \circ \pi^{\alpha}_{pf(p)}$ is zero for all $p \in \mathbb{Z}$. As for (ML), we say that $(M_{p,\alpha})_{p,\alpha}$ satisfies (ML), if, for every index α, there exist a $\beta \geq \alpha$ and a mapping $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ ($f(p) \geq p$) such that $\mathrm{Im}\,(\pi^{\beta}_{pq} \circ u^{q}_{\beta\alpha}) = \mathrm{Im}\,(\pi^{\beta}_{pf(p)} \circ u^{f(p)}_{\beta\alpha})$ for all p and $q \geq f(p)$. We need the following

<u>Lemma.</u> a) If the system $(M_{p,\alpha})_{p,\alpha}$ is essentially zero, then $\varinjlim_{\alpha} \varprojlim_{p} M_{p,} = 0$ and $\varinjlim_{\alpha} R^{1}\varprojlim_{p} M_{p,} = 0$.
b) If the system $(M_{p,\alpha})_{p,\alpha}$ satisfies (ML), then $\varinjlim_{\alpha} R^{1}\varprojlim_{p} M_{p,\alpha} = 0$.

Proof) We assume that $(M_{p,\alpha})_{p,\alpha}$ satisfies (ML) (resp. is essentially zero). It suffices to show that, for every index α, there exists a $\beta \geq \alpha$ such that the homomorphism

$$R^{i}\varprojlim_{p} u^{p}_{\beta\alpha} \ : \ R^{i}\varprojlim_{p} M_{p,\alpha} \longrightarrow R^{i}\varprojlim_{p} M_{p,\beta}$$

is zero for i = 1 (resp. i = 0, 1). For any index α, we take the $\beta \geq \alpha$ in the definition above, and set

$$N_{p} = \mathrm{Im}\,(u^{p}_{\beta\alpha} : M_{p,\alpha} \longrightarrow M_{p,\beta}).$$

Then the morphism of projective systems

$$(u^p_{\beta\alpha})_p \ : \ (M_{p,\alpha})_p \longrightarrow (M_{p,\beta})_p$$

is factored through the projective system $(N_p)_p$. Passing to the limit, the homomorphism $R^i\varprojlim_p u^p_{\beta\alpha}$ is also factored through $R^i\varprojlim_p N_p$. Since $(N_p)_p$ satisfies (ML) (resp. is essentially zero) in the sense of n° 2, we have $R^i\varprojlim_p N_p = 0$ for $i = 1$ (resp. $i = 0, 1$). Hence, $R^i\varprojlim_p u^p_{\beta\alpha} = 0$ for $i = 1$ (resp. $i = 0, 1$). q.e.d.)

Rewrite the conditions "essentially zero" and (ML) for the system $(L^i(K_\alpha)_p)_{p,\alpha}$ as we did in n° 2. Then we have a dictionary similar to (2.9) and (2.10) :

(3.2) $(K^\cdot_\alpha)_\alpha$ satisfies (B^*_i). $\Longleftrightarrow$ $(L^i(K_\alpha)_p)_{p,\alpha}$ is essentially zero.

(3.3) $(K^\cdot_\alpha)_\alpha$ satisfies (WB^*_i). $\Longleftrightarrow$ $(L^i(K_\alpha)_p)_{p,\alpha}$ satisfies (ML).

If $(K^\cdot_\alpha)_\alpha$ satisfies (WB^*_i), then the system $(L^i(K_\alpha)_p)_{p,\alpha}$ satisfies (ML), and we have $\varinjlim_\alpha R^1\varprojlim_p L^i(K_\alpha)_p = 0$ by Lemma above. Taking the inductive limit of the sequence (3.1), we get an isomorphism

$$\psi^i = \varinjlim_\alpha \psi^i_\alpha \ : \ H^i(\varinjlim_\alpha K^\wedge_\alpha) = \varinjlim_\alpha H^i(K^\wedge_\alpha) \xrightarrow{\sim} \varinjlim_\alpha \varprojlim_p H^i(K_\alpha/F^pK_\alpha).$$

This gives our result.

References

[1] A. Grothendieck : E.G.A., 0_{III}, §§11-13.

[2] R. Hartshorne : On the de Rham cohomology of algebraic varieties, Publ. Math. IHES, 45 (1976).

[3] N. Sasakura : Cohomology with P.G. and completion theory (in this vol