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Complexity of Some Strategies Proving Theorems in the Propositional Logic

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1. Introduction

The unsatisfiability problem or the satisfiability problem for propositional formulas is one of the most typical NP-complete problems [2]. It is also one of the most important open problems in the theory of computation to find out whether or not unsatisfiability or satisfiability for any given propositional formula can be decided in deterministic polynomial time of its size, that is, whether NP=P.

The resolution principle by J.A.Robinson or other complete resolutions [1] are powerful enough to decide unsatisfiability or satisfiability for any given propositional formula in set of clauses, but they necessitate more time than polynomial time of input size.

On the other hand, unsatisfiability of a horn set of propositional formulas can be decided by unit resolution or input resolution [3] and according to N.D.Jones et al. [4] it can be decided in deterministic polynomial time whether there is a unit (or an input) resolution refutation from any given propositional formula.

Unit resolution and input resolution were introduced by C.L.Chang [1].
In the propositional logic, unit resolution is an inference rule to derive a formula (called a resolvent) from a literal and a formula (a clause) containing its complementary literal.
Unit resolution is equivalent to input resolution, whose deduction takes a linear form in which center clauses are resolvents and side clauses are given input clauses. In an input resolution deduction 'center clauses' (resolvents) are prohibited from side clauses.
Thus the input resolution deduction is a linear deduction [1] with such strong restrictions.

In order to decide unsatisfiability for a more general class than the class of propositional horn sets in deterministic polynomial time, by relaxing the restrictions about side clauses in the input resolution, we formulated a Restricted Linear (RL) deduction, in which some center clauses can again be side clauses [5].

From the constructive viewpoint we found that the RL deduction is formed by layering linear deductions each of which corresponds to an input resolution refutation. Thus we call this type of RL deduction by another name: a Linear Layered Resolution deduction based on Input resolutions (an LLRI deduction).

Next, as an extension of LLRI, we formulate a nonlinear form of layered resolution deduction based on input resolutions (an NLRI deduction) in which resolvents are generated by LLRI from both input clauses and unit clauses obtained by LLRI deductions.

In this paper we propose a more extended version of layered resolutions based on input resolutions in which (1) linear deductions corresponding to more general input resolution refutations are layered and (2) resolvents containing a definite number of literals can be memorized and used for further resolution deductions. We apply it to the propositional logic.

Also we characterize a class of propositional sets of clauses which is more general than the class of Horn sets in [6] and for which the unsatisfiability problem is P-complete, on the basis of LLRI deductions.

2. A Restricted Linear Resolution (RL)— A Linear Layered Resolution Based on Input Resolutions (LLRI)

From now on we treat resolutions in the propositional logic. We briefly introduce the resolutions below.

Resolutions are inference rules applied to formulas in set of clauses. A clause is a disjunction or a set $L_1 \vee L_2 \vee \ldots \vee L_n$ of literals L_i, which are atomic formulas such as P, Q, R and so on to denote propositions or the negation of atomic formulas.

A set of clauses is equivalent to a conjunction among clauses.

In the propositional logic, the resolution principle introduced by J.A.Robinson is an inference rule to derive

$M_1 \vee M_2 \vee \ldots \vee M_i \vee N_1 \vee N_2 \vee \ldots \vee N_j$,

which is called a resolvent and denoted by $\mathrm{Res}(L \vee M_1 \vee M_2 \vee \ldots \vee M_i, \sim L \vee N_1 \vee N_2 \vee \ldots \vee N_j)$, from parent clauses $L \vee M_1 \vee M_2 \vee \ldots \vee M_i$ and $\sim L \vee N_1 \vee N_2 \vee \ldots \vee N_j$, where $\sim L$ is the negation of a literal L. L and $\sim L$ are called literals resolved upon. The empty clause □ is derived from L and $\sim L$ for any literal L.

Unit resolution in the propositional logic is an inference rule to derive

$M_1 \vee M_2 \vee \ldots \vee M_i$

from parent clauses L and $\sim L \vee M_1 \vee M_2 \vee \ldots \vee M_i$ for some literal L.

Input resolution is an inference rule to derive a clause from parent clauses in which one of the two parent clauses is an input clause, where the input clause is the one in a given set of clauses.

Definition 1 [1]:

Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, C_2, \ldots, C_k$ of clauses such that each C_i is either a clause in S or a resolvent of clauses preceding C_i, and $C_k=C$. A deduction of the empty clause from S is called a (resolution) refutation from S.

Theorem 1 [1]:

There is a unit resolution refutation from a set S of clauses if and only if there is an input resolution refutation from S.

Now we define a restricted linear deduction (for short, RL deduction), which will be called by another name: a linear layered resolution based on input resolutions (LLRI).

Definition 2:

Given a set S of clauses and a clause C_0 in S, an RL deduction of C_n with top clause C_0 is a deduction of the linear form of resolution in which:

1. For i=0, 1,..., n-1, C_{i+1} is a resolvent of C_i (called a center clause), and B_{i+1} (called a side clause), and each B_{i+1} is either in S, or in the set MC as defined in 2. C_n is in MC.
2. (the set MC)

(1) (i) C_0 is in MC.

(ii) If C_i is in MC and j (greater than i) is the least integer satisfying the following condition, then C_j is also in MC. C_j is said to be adjacent to C_i.

(2) (Condition)

(i) $C_i \not\supset C_j$.

(ii) The literals resolved upon in obtaining C_{i+1}, C_{i+2},..., C_j cannot be contained in C_j.

(iii) The literals in C_j have their terms unchanged (This condition is unnecessary in the propositional logic).

An RL deduction of the empty clause is called an RL (resolution) refutation.

Example 1:

Let $S=\{P \vee Q,\ P \vee \sim Q,\ \sim P \vee Q,\ \sim P \vee \sim Q\}$ be a set of clauses, where P and Q denote predicate symbols. There is no unit or input refutation from S, because no unit clause is in S.

An RL refutation from S is as follows:

$C_0=P \vee Q$, $C_1=Q$, $C_2=P$, $C_3=\sim Q$ and $C_4=\square$ are center clauses, and $B_1=\sim P \vee Q$,

$B_2=P\vee\sim Q$, $B_3=\sim P\vee\sim Q$ and $B_4=Q$ are side clauses, where C_{i+1} is a resolvent of C_i and B_i for i=0, 1, 2, 3, and C_0, C_1 and C_4 are in MC.

Next, fundamental properties of unit, input and RL deductions are shown.

Definition 3:

Let S be a set of clauses, and assume that there is a unit resolution refutation from S. Consider a deduction tree T for a unit resolution refutation from S in which nodes correspond to clauses obtained by unit resolution.

A clause C is said to be admissible to C' with respect to S if there is a path from the node for C to the node for C' in a deduction tree T, where the path should be traced in the direction from nodes for parent clauses to nodes for their resolvents.

Lemma 1:

Let S be a set of clauses in the propositional logic, and assume that there is a unit refutation from S. Assume that a literal L is resolved with $\sim L$ in the unit resolution refutation from S. Then

1. There is a clause C in S containing L such that there is a unit resolution refutation from the union of a set $S-\{C\}$ of clauses and a clause C-L.
2. There is an input resolution refutation with top clause C, in which C is not used as a side clause.

Using Lemma 1, we have the following theorem.

Theorem 2:

Let S be a set of clauses in the propositional logic, and assume that there is a unit resolution refutation from S. Then there is an input resolution refutation from S with any clause C_0, admissible to $\square$ with respect to S, as top clause. Also there is an RL resolution refutation from S.

Now we discuss another relation between unit and input resolutions.

Lemma 3:

Assume that there is an input resolution refutation from a set of clauses with top clause $L \cup C$ $(L \notin C)$.

Then (1) there is an input deduction of L with top clause $L \cup C$, in which L is not included in literals upon which center clauses are resolved, and (2) there is an input resolution refutation with top clause L.

Definition 4:

Let C_1 and C_2 be clauses such that $C_1 = C_2 \cup L$ for some literal L. Then C_2 is called to have a relation R_1 with C_1.

Applying Lemma 3 repeatedly, we can obtain the following lemma.

Lemma 4:

Let S be a set of clauses. Assume that there is an input resolution refutation with top clause C_0 from S.

Then there exist center clauses $C_1, C_2, \ldots, C_n = \square$ in the refutation such that C_{i+1} has a relation R_1 with C_i for $i=0, 1, \ldots, n-1$.

Utilizing Lemma 4, we can obtain the next result.

Theorem 5:

Let S be a set of clauses. Assume that there is an input resolution refutation with top clause C_0, then C_0 is admissible to □ with respect to S.

From the definition of the RL deduction, we have the following theorems.

Theorem 6:

Assume that there is an RL deduction in which C_j is adjacent to C_i and $C_k=Res(C_{k-1},B_k)$ for k such as $i+1\leq k\leq j$. Then there is an input resolution refutation with top clause $C_i'=C_i-C_j$ and side clauses $B_k'=B_k-C_j$ for k between i+1 and j in the order.

Theorem 7:

Assume that a linear deduction of C_j with top clause C_i such that

1. $C_{k+1}=Res(C_k,B_{k+1})$ for k such as $i\leq k\leq j-1$.
2. B_{k+1} is an input clause or C_i for k between i and j-1.
3. There is an input resolution refutation from C_i', B_{i+1}',..., B_j' for $C_i'=C_i-C_j$ and $B_k'=B_k-C_j$ $(i+1\leq k\leq j)$.

Then the linear deduction is an RL deduction.

Thus an RL deduction is formed by layering linear deductions each of which corresponds to an input resolution refutation. Thus we call an RL deduction by another name: a Linear Layered Resolution deduction based on Input resolutions (an LLRI deduction).

It is obvious that an input resolution refutation is also an RL resolution refutation.

Lemma 8:

Assume that there is an input resolution refutation from S with top clause C_0, and that $C_0 \subset C_1$ in S. Then there is an input resolution refutation from $S-\{C_1\}$ with top clause C_0.

From Theorems 6 and 7, and Lemma 8, we have the theorem:

Theorem 9:

Assume that an RL (LLRI) deduction of C_j and C_j is adjacent to C_i. Then there is an RL deduction of C_j with top clause C_i in which any clause in MC except C_i is not used as a side clause.

3. RL (LLRI) Refutability for the Propositional Logic

From Theorems 6 and 7, and Lemma 4, we have the theorem:

Theorem 10:

Assume that there is an RL (LLRI) deduction of C_j with top clause C_i. Then there is an RL deduction of C_j with top clause C_i such that any clause in MC between C_i and C_j has a relation R_1 with its adjacent clause.

It is concluded from Theorems 6 , 7 and 10 that there is some input resolution refutation with the specified literal as top clause for each RL deduction of a clause from an RL refutable set of clauses with its adjacent one as top clause.

Thus a decision algorithm (Algorithm 2) of RL resolution refutability from a given set of clauses can be constructed by using a decision algorithm (Algorithm 1) of input resolution refutability which can be decided by unit resolution.

By Theorems 2 and 5, input resolution refutability from a given set of clauses with a literal L as top clause can be decided by the algorithm to determine whether there is a unit resolution refutation from the set in which the literal L is admissible to □ with respect to the set.

Algorithm 1 is constructed by gathering and storing the literals which are generated from the specified literal or its resolvents by unit resolution and contribute to derivation of the empty clause.

Utilizing Algorithm 1 repeatedly, we construct Algorithm 2 to decide an RL resolution refutability from a set of clauses.

Algorithm 1: (A decision algorithm of an input resolution refutability with top clause L from a set S of clauses)

Input: $S=\{S_1,S_2,\ldots,S_m\}$, a set of clauses in the propositional logic, and a literal L in S.

Output: 'Yes', if there is an input refutation from S with top clause L. 'No', otherwise.

Method: The algorithm is as follows.

Let T be a set of literals. T initially consists of the literal L. Let U be a set of literals. U is initially empty. V is an auxiliary boolean variable indicating whether new literals are generated or not.

```
Procedure Unit(S)
begin
    V←1;  T←L;  U←φ;
    while V=1 do
        for each S_i do
            for each L_ij in S_i do
                if L_ij ∉ T∪U
                then
                begin
                    Decide whether there is a unit deduction of L_ij
                    from S;
                    if a deduction exists and some literal in T is
                       used
                    then
                    begin
                        T← T∪L_ij;  V←1
                    end;
                    if a deduction exists and no literal in T is used
                    then
                    begin
                        U← U∪L_ij;  V←1
                    end;
                    V←0
                end
                else V←0;
    if ~L exists in T∪U for some L in T  then return 'Yes'
    else return 'No'
end
```

Algorithm 2: (A decision algorithm of an RL resolution refutation from a set S of clauses)

Input: $S=\{S_1,S_2,\dots,S_m\}$.

Output: 'Yes', if there is an RL resolution refutation from S.
'No', otherwise.

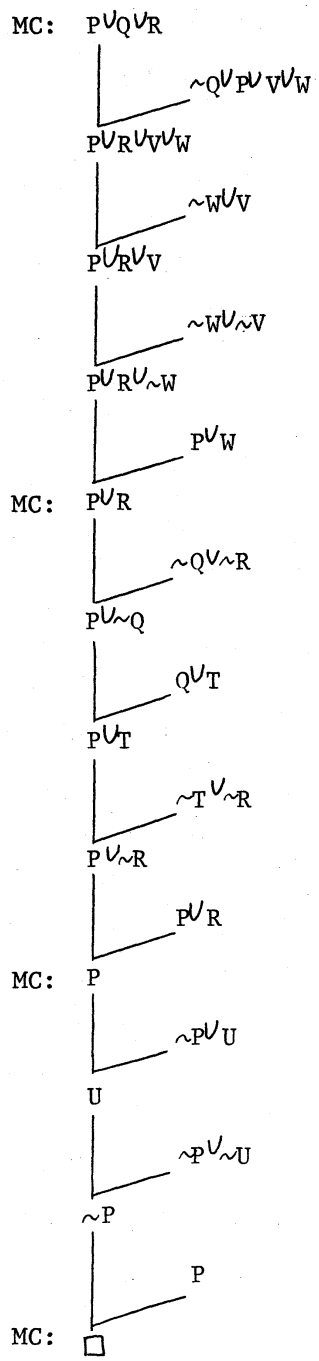
Method: The algorithm is as follows. V is an auxiliary boolean variable indicating whether RL deductions exist.

```
Procedure RL(S):
begin
    U←φ;  i=1;
    while i≤m do
        begin
            V←1;
            while V=1 do
                begin
                    for each Lij∈Si-U do
                        begin
                            Decide whether there is an input reso-
                            lution refutation with top clause Lij
                            from {S1-U,S2-U,...,Sm-U}  (Algorithm 1);
                            if an input resolution refutation exists
                            then begin U←U∪Lij;  V←1 end
                            else V←0
                        end;
                    if U=Si then return 'Yes'
                end;
            U←φ;  i=i+1
        end;
    return 'No'
end
```

<u>Example 2</u>:

Let $S=\{P\vee Q\vee R,\ \sim Q\vee P\vee V\vee W,\ \sim W\vee V,\ \sim W\vee\sim V,\ P\vee W,\ \sim Q\vee\sim R,\ Q\vee T,\ \sim T\vee\sim R,\ \sim P\vee U,\ \sim P\vee\sim U\}$ be a set of clauses, where P, Q, R, T, U, V and W denote atomic formulas (propositions).

There is an RL refutation from S, if we assume Definition 2. But there is no RL refutation from S, if we take Definition 15 in [5].



Complexity of Algorithms 1 and 2 by means of multitape Turing machines is as follows.

Theorem 11:

Algorithm 1 is of time complexity $O(n^2)$ and of space complexity $O(n)$ for an input set of length n.

Theorem 12:

Algorithm 2 (a decision algorithm of an RL resolution refutability from a set of clauses) is of time complexity $O(n^4)$ and of space complexity $O(n)$ for an input set of length n.

4. A Nonlinear Layered Resolution Based on Input Resolutions (NLRI)

Here we formulate a nonlinear (form of) deduction by extending the LLRI deduction in such a way that clauses which consist of a definite number of literals and are obtained by LLRI deductions can be used as side clauses of other LLRI deductions.

Definition 5:

We define a Nolinear (form of) Layered Resolution based on Input resolutions (NLRI) as follows (this is an extension of RL1 in [5]):

If C is a clause containing not greater than k literals, an RL (LLRI) deduction of C from $\{S_1, S_2, \ldots, S_m\}$ is denoted by

$k\text{-LLRI}(S_1, S_2, \ldots, S_m : C)$.

A k-NLRI deduction of C from $S=\{S_1, S_2, \ldots, S_m\}$ for C being a clause containing not greater than k literals is defined recursively in the following and is denoted by

$k\text{-NLRI}(S_1, S_2, \ldots, S_m : C)$.

1. $k\text{-LLRI}(T_1, T_2, \ldots, T_m : C)$ for T_i in S is also $k\text{-NLRI}(T_1, T_2, \ldots, T_m : C)$.

2. If there are

   k-LLRI$(T_1,T_2,\ldots,T_k:L)$ for a literal L and T_i in S, and

   k-LLRI$(U_1,U_2,\ldots,U_j,L:C)$ for U_i in S,

   then there is

   k-NLRI$(T_1,T_2,\ldots,T_k,U_1,U_2,\ldots,U_j:C)$.

3. The deduction defined by 1 and 2 recursively is a k-NLRI deduction.

4. All k-NLRI deductions are defined by applying the above rules.

We can construct an algorithm to decide an NLRI resolution refutability from a set S of clauses in two steps as below.

<u>Algorithm 3</u>: (An algorithm to decide a k-NLRI resolution refutability from S)

1. For a literal selected from S, decide whether there is an LLRI deduction of the literal, and if such a deduction exists, add it to an input set (by Algorithm 4).

   Repeat such a decision until such literals are exhausted.

2. Decide whether there is an LLRI resolution refutation from the input set and the added literals.

An algorithm to decide whether there is a k-LLRI deduction of a literal L from an input set (Step 1 of Algorithm 3) is given as follows.

<u>Algorithm 4</u>: (An algorithm to decide whether there is a k-LLRI deduction of a literal L from an input set S)

Let $S=\{S_1,S_2,\ldots,S_m\}$.

Apply Algorithm 2 for a set of clauses

$S'=\{S_1-L,S_2-L,\ldots,S_m-L\}$, where

S_i-L should not be selected as top clause if $S_i-L=S_i$.

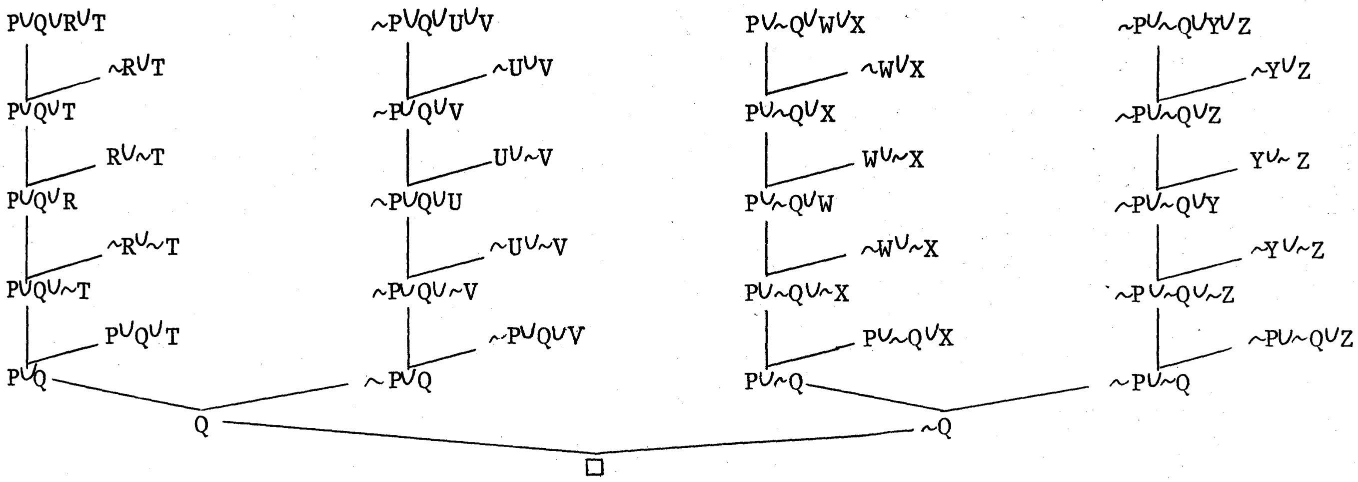
If an LLRI refutation from S' exists, then a k-LLRI deduction of L from S exists.

If no LLRI refutation from S' exists, then no k-LLRI deduction of L from S exists.

Example 2:

Let $S=\{P\vee Q\vee R\vee T,\ \sim R\vee T,\ R\vee\sim T,\ \sim R\vee\sim T,\ \sim P\vee Q\vee U\vee V,\ \sim U\vee V,\ U\vee\sim V,$ $P\vee\sim Q\vee W\vee X,\ \sim W\vee X,\ W\vee\sim X,\ \sim W\vee\sim X,\ \sim P\vee\sim Q\vee Y\vee Z,\ \sim Y\vee Z,\ Y\vee\sim Z,\ \sim Y\vee\sim Z\}$ be a set of clauses, where P, Q, R, T, U, V, W, X, Y and Z are propositions.

There is a 2-NLRI refutation from S, although there is no 1-NLRI refutation.



Eventually, we can obtain the following theorem.

Theorem 13:

Algorithm 3 (An algorithm to decide an NLRI resolution refutability) is of time complexity $O(n^{4k+2})$ and of space complexity $O(n^k)$ for an input set of length n.

5. A Class of Extended Horn Sets

A Horn set is a set in which no clause contains more than one positive literal.

According to [3], the following theorems hold in the propositional logic.

Theorem 14 [3]:

If S is an unsatisfiable Horn set, then there exists an input refutation from S.

Definition 6:

A set S of clauses is minimally unsatisfiable if S is unsatisfiable and no proper subset of S is unsatisfiable.

Theorem 15 [3]:

Let S be a minimally unsatifiable, input refutable set of clauses, then there exists a set S' such that S' is a renaming of S and S' is a Horn set.

We extend these results with respect to LLRI.

Definition 7:

Let C be a set of positive literals. C-Horn set is a set of clauses each of which contains all the literals in C and at most one positive literal not in C (□-Horn set is exactly a Horn set).

Definition 8:

Let S be a set of clauses. S is said to be well-partitioned by linearly ordered extended Horn sets if there are C_i-Horn sets for $1 \leq i \leq n$ such that:

1. Each clause in S belongs to at least one of C_i-Horn sets.
2. $C_i \supsetneq C_{i+1}$ for each i.
3. $C_n = \Box$.

We can obtain the following results.

Theorem 16:

Let S be an unsatisfiable set well-partitioned by linearly ordered extended Horn sets, then there is an LLRI refutation from S.

Theorem 17:

It can be decided in $O(n^3)$ time and in $O(n)$ space to decide whether a given set of length n can be well-partitioned.

Definition 9:

A set S of clauses is minimally LLRI refutable if there is an LLRI refutation from S and there is no LLRI refutation from any proper subset of S.

Theorem 18:

If S is a minimally LLRI refutable set of clauses, then there exists a set S_1 such that:

1. S_1 is well-partitioned by linearly ordered extended Horn sets.
2. $S \cup S_0$ implies S_1, where S_0 is a set of clauses each of which contains two literals.

According to [4] and [3], it can be concluded that the unsatisfiability problem for Horn sets in the propositional logic is P-complete.

Consider the transformation for a Horn set $S=\{C_1, C_2, \ldots, C_m\}$

TF1: $S \to S' = \{ C_1 \vee \sim L, C_2 \vee \sim L, \ldots, C_m \vee \sim L,\ L \vee L_1 \vee L_2,\ \sim L_1 \vee L_2,\ L_1 \vee \sim L_2,\ \sim L_1 \vee \sim L_2 \}$,

where L, L_1 and L_2 are atoms not appearing in S.

Obviouly S' is well-partitioned by $\{L \vee L_1\}$-Horn set and $\Box$-Horn set. S is unsatisfiable if and only if S' is unsatisfiable. Thus we obtain the next theorem.

Theorem 19:

The unsatisfiability problem for propositional sets well-partitioned by linearly ordered Horn sets is P-complete.

6. Concluding Remarks

In this paper we defined an extended version of layered resolutions based on input resolutions proposed in [5] which are extensions of unit and input resolutions. We applied the resolutions to the propositional logic.

We extended the restricted linear resolution deduction proposed in [5] such that it is formed by layering linear deductions each of which corresponds to a more general input refutation and called it by another name: a linear layered resolution deduction based on input resolutions (LLRI deduction).

Next we formulated a nonlinear layered resolution deduction based on input resolutions (NLRI deduction), as an extension of the LLRI and RL1 in [5].

On the basis of LLRI, we characterized a class of propositional sets of claused which is more general than the class of Horn sets and for which the unsatisfiability problem is P-complete.

It is left for future study to characterize a class of propositional sets of clauses for which the unsatisfiability problem is P-complete, on the basis of NLRI.

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