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SOME ASPECTS OF WEIGHTED AND NON-WEIGHTED HARDY SPACES

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The main aim of this talk is to present some results on weighted and non-weighted Hardy spaces. We also put some emphasis on the interrelations between these spaces and with other spaces existing in the literature, such as Besov (Lipschitz) spaces and Triebel-Lizorkin spaces. These interrelations are marked by the fact that the study of function spaces in recent years has flourished due partially to the combination of the technique of maximal functions developed by Fefferman-Stein and the spectral decomposition of Peetre.

§1. H^p and h^p spaces

We begin by recalling an equivalent definition for the space H^p as given by Fefferman-Stein [1]. A tempered distribution f on R^n is said to be in H^p, $0 < p < \infty$, if the non-tangential maximal function $N(f)(x) = \sup_{|x-y|<t} |\varphi_t * f(y)| \in L^p$, where $\varphi \in S$ with $\int \varphi(x)dx = 1$ and $\varphi_t(y) = t^{-n}\varphi(y/t)$. We put $||f||_{H^p} = ||N(f)||_p$. There are various characterizations of H^p by using other maximal functions or harmonic functions on R_+^{n+1}. We refer to [1] for details. It is known that $H^p = L^p$ for $1 < p < \infty$, and H^p $(0 < p \leq 1)$ are good substitutes for L^p from a number of points of view. However, these spaces break down at some other points, such as S is not entirely contained in H^p and pseudo-differential operators are not bounded on H^p $(0 < p \le 1)$. To remedy these facts, Goldberg [2] has considered the space h^p which can be defined as follows. A tempered distribution f is in h^p, $0 < p < \infty$, if $N(f)(x) = \sup_{|x-y|<t<1/2} |\varphi_t * f(y)| \in L^p$, where φ is as above. (There should be no confusion about the notation N(f) since if will be clear from the context whether we are dealing with H^p or h^p; similar abuses of the notation will be used in the rest of this note.) Further, we put $\|f\|_{h^p} = \|N(f)\|_p$.

Before proceeding on, we need some notations. Let S denote the strip domain $R^n \times]0,1[$. The Poisson kernel for S, denoted by P, is given by $P(x, t) = P_t(x) = P^0(x, t) + P^1(x, t)$, where $\hat{P}^0(\xi, t) = \sinh\{(1 - t)2\pi|\xi|\}/\sinh\{2\pi|\xi|\}$, $P^1(x, t) = P^0(x, 1 - t)$, and $\hat{f}$ stands for the Fourier transform of $f \in L^1$. Since $P_t \in \mathcal{S}$, $P_t * f = Pf$ is a well-defined harmonic function on S for each $f \in \mathcal{S}'$. The main results on h^p are summarized in the next theorem (see [2]). We use I to denote a cube with sides parallel to coordinate axes hereafter. All immaterial constants are denoted by $C, C_1, \ldots, c, c_1, \ldots$. They are not necessarily the same on any two occurrences.

<u>THEOREM A</u>. (i) A tempered distribution f is in h^p $(0 < p < \infty)$ if and only if $N(u)(x) = \sup_{|x-y|<t<1/2} |u(y, t)| \in L^p$, where $u = Pf$. Further, one has $\|f\|_{h^p} \approx \|N(u)\|_p$.

(ii) $\mathcal{S}$ is dense in h^p and pseudo-differential operators of class $S_{0,1}$ are bounded on h^p $(0 < p < \infty)$.

(iii) Let $\psi \in \mathcal{S}$ so that $\int \psi(x)dx = 1$ and $\int x^\alpha \psi(x)dx = 0$ for every $|\alpha| \ne 0$, and $f \in h^p$. Then $f - \psi * f \in H^p$ and $\|f - \psi * f\|_{H^p} \le C\|f\|_{H^p}$.

(iv) $f \in h^1$ if and only if f and $r_j^{\varphi} f$ $(j = 1, \ldots, n)$ are in L^1, where $\varphi \in S$, $\varphi = 1$ on a neighborhood of the origin and $(r_j^{\varphi} f)^{\wedge} = -(1 - \varphi) i(\xi_j/|\xi|)\hat{f}$. Further,

$$\|f\|_{h^1} \approx \|f\|_1 + \sum_{j=1}^{n} \|r_j^{\varphi} f\|_1.$$

(v) $$(h^p)^* = B_{\infty,\infty}^{n/p-n} \qquad (0 < p < 1),$$

$$(h^1)^* = \mathrm{bmo}.$$

Here $B_{\infty,\infty}^{n/p-n}$ is the Besov (Lipschitz) space defined by Taibleson [3] and bmo is the space of all locally integrable functions b such that $\sup_{|I|<1} (1/|I|) \int_I |b(x) - b_I| dx < \infty$ and $\sup_{|I|\geq 1} (1/|I|) \int_I |b(x)| dx < \infty$, where $b_I = (1/|I|) \int_I b(x) dx$. ($B_{\infty,\infty}^{n/p-n}$ was denoted by $\Lambda(n/p-n; \infty, \infty)$ by Taibleson.)

(vi) A distribution f is in h^p $(0 < p \leq 1)$ if and only if there exist a sequence $\{a_j\}$ of (h^p, ∞)-atoms and a sequence $\{\lambda_j\}$ such that $f = \Sigma \lambda_j a_j$ in S' and $\Sigma |\lambda_j|^p < \infty$. (A function a, supported in a cube I, is called an (h^p, ∞)-atom if $\|a\|_\infty \leq |I|^{-1/p}$, and $\int_I x^\alpha a(x) dx = 0$ for all $|\alpha| \leq [n/p-n]$ if $|I| < 1$.)

(vii) If α is a C^∞-diffeomorphism of R^n onto R^n such that $\alpha(x) = x$ for $|x| \geq 1$, and $f \in h^p$ $(0 < p < \infty)$. Then $\|f \circ \alpha\|_{h^p} \leq c\|f\|_{h^p}$.

Next we give definitions for Besov spaces and Triebel-Lizorkin spaces by using the spectral decomposition of Peetre ([4], [5], [6], [7]). Let φ be a function in S which satisfies the following properties:

(a) supp $\varphi = \{1/2 \le |\xi| \le 2\}$ and $\varphi(\xi) > 0$ on $\{1/2 < |\xi| < 2\}$,

(b) $$\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1 \quad \text{for} \quad |\xi| \neq 0.$$

Let φ_k, $k = 0, 1, \ldots$, be the functions in S given by

$$\hat{\varphi}_k = \varphi(2^{-k}\xi), \ k = 1, 2, \ldots,$$

$$\hat{\varphi}_0 = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi).$$

The homogeneous and non-homogeneous Besov spaces and Triebel-Lizorkin spaces are defined as follows.

$$B^s_{p,q} = \left\{ f \in S';\ \|f\|_{B^s_{p,q}} = \left\{ \sum_{k=0}^{\infty} [2^{ks} \|\varphi_k * f\|_p]^q \right\}^{1/q} < \infty \right\},$$

$$\dot{B}^s_{p,q} = \left\{ f \in S';\ \|f\|_{\dot{B}^s_{p,q}} = \left\{ \sum_{k=-\infty}^{\infty} [2^{ks} \| F^{-1}(\varphi(2^{-k}\xi)\hat{f}) \|_p]^q \right\}^{1/q} < \infty \right\},$$

$$F^s_{p,q} = \left\{ f \in S';\ \|f\|_{F^s_{p,q}} = \left(\int \left[\sum_{k=0}^{\infty} 2^{skq} |\varphi_k * f(x)|^q \right]^{p/q} dx \right)^{1/p} < \infty \right\},$$

$$\dot{F}^s_{p,q} = \left\{ f \in S';\ \|f\|_{\dot{F}^s_{p,q}} \right.$$

$$\left. = \left(\int \left[\sum_{k=-\infty}^{\infty} 2^{ksq} | F^{-1}(\varphi(2^{-k}\xi)\hat{f})(x) |^q \right]^{p/q} dx \right)^{1/p} < \infty \right\},$$

where $-\infty < s < \infty$, $0 < p, q \le \infty$ and F^{-1} denotes the inverse Fourier transform.

It is known that

$$H^p = \dot{F}^0_{p,2} \quad (0 < p < \infty). \tag{1}$$

(See [6] or [7]). Professor Triebel suggested to the author that

$$h^p = F^0_{p,2} \quad (0 < p < \infty). \tag{2}$$

The proof of (2) is based on an idea used by Peetre and Triebel ([6], [7]) in the proof of (1). We shall use the following Hilbert space version of a result of Goldberg on multipliers of the spaces h^p. Let $\{f_k\}$ be a sequence in S. Put

$$\|\{f_k\}\|_{h^p(\ell_2)} = \left\| \sup_{0<t<1} \left[\sum |\Phi_t * f_k(x)|^2\right]^{1/2} \right\|_p ,$$

where Φ is a function in S whose Fourier transform $\hat{\Phi}$ is of compact support and $\int \Phi(x)dx = 1$.

<u>LEMMA 1</u> (Goldberg [2]). (i) If $m \in C^\infty$ and $(1 + |x|^2)^{|\alpha|/2} |D^\alpha m(x)| \le C_\alpha$ for all α, and $\hat{K} = m$. Then $\|K*f\|_{h^p} \le C\|f\|_{h^p}$.

(ii) If $\{m_k\}$ is a sequence in C^∞ such that $(1 + |x|^2)^{|\alpha|/2} [\Sigma |D^\alpha m_k(x)|^2]^{1/2} \le C_\alpha$ for all α, and $\hat{K}_k = m_k$, then

$$\|\Sigma_k K_k * f_k\|_{h^p} \le C \|\{f_k\}\|_{h^p(\ell_2)} .$$

We shall also need a maximal inequality for $F^s_{p,q}$.

<u>LEMMA 2</u> (cf. [4], [7]). Let $0 < p < \infty$, $0 < q \le \infty$, $-\infty < s < \infty$ and f be a distribution in $F^s_{p,q}$. Then

$$\|\{2^{sk}\varphi_k^* f\}\|_{L^p(\ell_q)} \le C\|f\|_{F^s_{p,q}} ;$$

where for each k = 0, 1, 2,...

$$\varphi_k^* f(x) = \sup_{y \in R^n} \frac{|\varphi_k * f(x - y)|}{1 + 2^{k\lambda}|y|^\lambda} , \quad x \in R^n$$

and λ is a sufficiently large positive number that depends on n, p and q.

REMARK 1. Results similar to those in Lemma 2 hold for the spaces $B^s_{p,q}$, $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ ([4], [7]). This kind of maximal inequality was first introduced by Peetre in the study of $F^s_{p,q}$ in the full range $0 < p, q \le \infty$ ([4]). We also note that the proof of Lemma 2 is to some extent modelled after [1; pp. 183-187].

PROOF OF (2). Since it is known that $\mathcal{S}$ is dense in both h^p and $F^0_{p,2}$, it is enough to prove that

$$\|f\|_{h^p} \approx \|f\|_{F^0_{p,2}} \quad \text{for } f \in \mathcal{S}. \tag{3}$$

Let $\{r_m\}$ be the set of Rademacher functions (cf [8]), and let k be a positive integer. Keeping k fixed for a moment. Since $\Sigma^k_{m=0} r_m(t)\varphi_m$ satisfies the condition in (i) of Lemma 1 with the constants C_α independent of k and t, it follows from an identity for Rademacher functions [8; Appendix 4] and Lemma 1 that

$$\int\left[\sum_{m=0}^{k} |\varphi_m * f(x)|^2\right]^{p/2} dx \le C_1 \int\int_0^1 \left|\left(\sum_{m=0}^{k} r_m(t)\varphi_m\right) * f(x)\right|^p dt\,dx$$

$$\le C_2 \int_0^1 \left\|\left(\sum_{m=0}^{k} r_m(t)\varphi_m\right) * f\right\|^p_{h^p} dt \le C_3 \|f\|^p_{h^p}$$

Letting $k \to \infty$, we obtain

$$\|f\|_{F^0_{p,2}} \le C\,\|f\|_{h^p}. \tag{4}$$

To prove the converse, take a function ψ in $\mathcal{S}$ such that supp $\psi \subset \{1/3 \le |\xi| \le 3\}$ and $\psi = 1$ on $\{1/2 \le |\xi| \le 2\}$. Put $\hat{\psi}_k = \psi(2^{-k}\xi)$, $k = 1, 2, \ldots$, and let $\psi_0 \in \mathcal{S}$ be so that $\hat{\psi}_0 = 1$ on $\{|\xi| \le 2\}$. Then Lemma 1 implies that

$$\|f\|_{h^p} = \left\|\sum_{k=0}^{\infty} \psi_k * (\varphi_k * f)\right\|_{h^p} \le C_4 \|\{\varphi_k * f\}\|_{h^p(\ell_2)}$$

$$\leq C_5 \left\| \left[\sum_{k=0}^{\infty} \sup_{0<t<1} |\Phi_t * \varphi_k * f|^2 \right]^{1/2} \right\|_p . \tag{5}$$

Next we observe that

$$\Phi_t * \varphi_k * f = F^{-1}[\hat{\Phi}(t\cdot)\hat{\varphi}_k \hat{f}].$$

Hence, by comparing the support of $\hat{\Phi}(t\cdot)$ and $\hat{\varphi}_k$, we see that

$$\Phi_t * \varphi_k * f = 0 \text{ unless } t \leq c2^{-k}$$

for any $k = 1, 2, \ldots$. This in turn implies that for each $k = 0, 1, 2, \ldots$

$$\sup_{0<t<1} |\Phi_t * \varphi_k * f(x)| \leq \sup_{0<t\leq c2^{-k}} \int |\Phi(y)| |\varphi_k * f(x-ty)| dy$$

$$\leq C_6 \sup_{z\in R^n} \frac{|\varphi_k * f(x-z)|}{1 + 2^{k\lambda}|z|^{\lambda}} = C_6 \varphi_k^* f(x).$$

Therefore it follows from Lemma 2 that

$$\left\| \left[\sum_{k=0}^{\infty} \sup_{0<t<1} |\Phi_t * \varphi_k * f(x)|^2 \right]^{1/2} \right\|_p \leq C_6 \, \|\{\varphi_k^* f\}\|_{L^p(\ell_2)}$$

$$\leq C_7 \, \|f\|_{F^0_{p,2}} .$$

This, combined with (5), completes the proof of (2).

After (1) and (2) are established, the connection between Hardy spaces and Besov spaces are easily derived. In particular, we have the following inclusion relations:

$$\dot{B}^0_{p,p} \subset H^p \subset \dot{B}^0_{p,2} \tag{6}$$

and

$$B^0_{p,p} \subset h^p \subset B^0_{p,2} \tag{7}$$

if $0 < p \leq 2$.

$$\dot{B}^0_{p,2} \subset H^p \subset \dot{B}^0_{p,p} \tag{8}$$

and

$$B^0_{p,2} \subset h^p \subset B^0_{p,p} \tag{9}$$

if $2 \le p < \infty$.

Finally, we end this section by stating some results on interpolation. The Fefferman-Riviere-Sagher interpolation theorem asserts that

$$(H^p, H^q)_{\theta,r} = H^r, \tag{10}$$

$0 < p, q < \infty$, $0 < \theta < 1$ and $1/r = (1-\theta)/p + \theta/q$ (see [9]). By using (10), it can be proved that

$$(h^p, h^q)_{\theta,r} = h^r, \tag{11}$$

where p, q, θ and r are as above (see [7]).

§2. The spaces h^p_w

We first recall some facts about weight functions. Hereafter w is always assumed to be a non-negative locally integrable function on R^n.

A weight function w is said to be in A_p $(1 < p < \infty)$ if it satisfies

$$(A_p) \qquad \left\{\frac{1}{|I|}\int_I w(x)\,dx\right\}\left\{\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right\}^{p/p'} \le C$$

for all cubes I in R^n, where C is a constant independent of I and $1/p + 1/p' = 1$.

A weight functions w is said to be in A_∞ if there exist $c > 0$ and $r \ge 1$ such that

$$(A_\infty) \qquad |E| \le \alpha|I| \text{ implies } w(E) \le c\alpha^{1/r}w(I)$$

for any cube I in R^n and any measurable subset E of I, where $w(E) =$

$\int_E w(x)\,dx$.

A weight function w is said to be in A_1 if there exists $c > 0$ such that

$$(A_1) \qquad \frac{1}{|I|}\int_I w(x)\,dx \le c \operatorname{ess\,inf}_I w$$

for all cubes I in R^n. The condition $w \in A_1$ is equivalent to $w^*(x) \le cw(x)$ for almost every x, where f* stands for the (Hardy) maximal function of the locally integrable function f.

Basic properties of weight functions can be found in [10] and [11].

Let w be in A_∞ and $0 < p < \infty$. A function u, harmonic on S and symmetric with respect to $t = 1/2$, is said to be in h^p_w if $N(u)(x) = \sup_{\Gamma(x)} |u(y,t)| \in L^p_w$, where $\Gamma(x) = \{(y,t) \in S;\ |x-y| < t < 1/2\}$ and L^p_w denotes the space of all measurable functions g for which $\|g\|_{p,w} = \left\{\int |g(x)|^p w(x)\,dx\right\}^{1/p} < \infty$. The space h^p_w is equipped with the norm $\|u\|_{h^p_w} = \|N(u)\|_{p,w}$. This is a true norm only if $1 \le p < \infty$; however, we use this abuse of language for the sake of convenience.

We say that $F = (u, u_1, \ldots, u_n)$ is a Cauchy-Riemann system (in the sense of Stein-Weiss) if u and u_j $(j = 1, \ldots, n)$ are harmonic on S, and

$$\sum_{j=0}^{n} (\partial/\partial x_j) u_j = 0 \text{ on } S,$$

$$(\partial/\partial x_j) u_i = (\partial/\partial x_i) u_j \text{ on } S,\ i,\ j = 0, \ldots, n,$$

where we put $u_0 = u$ and $x_0 = t$. Further, in the rest of the paper we shall always assume that $u(x, t) = u(x, 1-t)$ and $u_j(x, t) = -u_j(x, 1-t)$ for any $j = 1, \ldots, n$ and for all $(x, t) \in S$ when we consider a Cauchy-Riemann system $F = (u, u_1, \ldots, u_n)$.

As usually happened in the study of weighted spaces (cf. [12], [13]), the behaviors of functions in h^p_w as $|x| \to \infty$ are a little worse than in the non-weighted case. This fact can be remedied by taking the weight w in an appropriate class and using the following harmonic majorization principle for subharmonic functions on S.

LEMMA 3 ([14; Theorem 2]). Let u be a real-valued function defined on $\overline{S} = R^n \times [0,1]$, subharmonic on S, and such that $\limsup_{(z,t)\in S,\ (z,t)\to(x,\delta)} u(z, t) = u(x, \delta)$ for every $x \in R^n$ and $\delta = 0, 1$. Further, assume that $\int (|u(x, 0)| + |u(x, 1)|)e^{-\pi|x|}dx < \infty$ and $\limsup_{|x|\to\infty} u^+(x, t)e^{-\pi|x|}|x|^{(n-1)/2} = 0$ uniformly in t, $0 < t < 1$. Then

$$u \leq P_t^0 * u(\cdot, 0) + P_t^1 * u(\cdot, 1) \text{ on } S.$$

Equality holds if u is furthermore assumed to be harmonic on S and continuous on $\overline{S}$.

Similar results hold if S is replaced by any of its substrip.

The basic properties of the spaces h_w^p are given in the next theorem.

THEOREM 1. (i) If $(n-1)/n < p < \infty$ and $w \in A_{pn/(n-1)}$, then $u \in h_w^p$ if and only if there exists a Cauchy-Riemann system $F = (u, u_1, \ldots, u_n)$ with the property that

$$||F||_{p,w} = \sup_{0<t<1} \left\{ \int \left(\sum_0^n |u_j(x, t)|^2 \right)^{p/2} w(x)dx \right\}^{1/p} < \infty.$$

Further, one has $||N(u)||_{p,w} \approx ||F||_{p,w}$.

(ii) If $w \in A_1$, then $u \in h_w^1$ if and only if there exists $f \in L_w^1$ such that $r_j f \in L_w^1$, $j = 1, \ldots, n$ and $u = Pf$, where $(r_j f)^\wedge = -i(\xi_j/|\xi|) \tanh\pi|\xi|\hat{f}$. Moreover, $||N(u)||_{1,w} \approx ||f||_{1,w} + \sum_1^n ||r_j f||_{1,w}$.

The proof of Theorem 1 is modelled after [12] and [13], and is based on the following two lemmas.

LEMMA 4. Let Φ be a continuous non-decreasing function on $[0,\infty[$ such that $\Phi(0) = 0$ and $\Phi(2\lambda) \le c_1\Phi(\lambda)$ for all $\lambda > 0$. Let $w \in A_\infty$ and u be a harmonic function on S such that $u(x, t) = u(x, 1-t)$ for $(x, t) \in S$ or $u(x, t) = -u(x, 1-t)$ for $(x, t) \in S$. Then there exist c and C, not depending on u, such that

$$c\int \Phi(N^0(u)(x))w(x)dx \le \int \Phi(A(u)(x))w(x)dx$$
$$\le C\int \Phi(N(u)(x))w(x)dx,$$

where

$$A(u)(x) = \left\{\iint_{\Gamma(x)} t^{1-n}\left(\sum_0^n |(\partial/\partial x_j)u(y, t)|^2\right)dydt\right\}^{1/2}$$

and

$$N^0(u)(x) = \sup_{\Gamma(x)} |u(y, t) - u(y, 1/2)|.$$

LEMMA 5. Let u be a harmonic function in h^p_w, $(n-1)/n < p < \infty$, $w \in A_{pn/(n-1)}$ and F be the Cauchy-Riemann system associated with u as in (i) of Theorem 1. Then there exists $F(\cdot, 0) \in L^p_w \times \cdots \times L^p_w = (L^p_w)^{n+1}$ such that $F(x, t) \to F(x, 0)$ almost everywhere and in $(L^p_w)^{n+1}$ as $t \to 0$. Further,

$$N(|F|)(x) \le C[(|F(\cdot, 0)|^{(n-1)/n})^*(x)]^{n/(n-1)}$$

for all $x \in R^n$, and $\|F\|_{p,w} \approx \|F(\cdot, 0)\|_{p,w}$. (If $n = 1$, then the number $(n-1)/n = 0$ in the conclusion of the lemma should be replaced by any q such that $p > q > 0$ and $w \in A_{p/q}$.)

REMARK 2. By (ii) of Theorem 1, we can identify h^1_w $(w \in A_1)$ with the space of all L^1_w-functions f for which $\|f\|_{h^1_w} = \|f\|_{1,w} + \Sigma_1^n \|r_j f\|_{1,w} < \infty$. We shall frequently use this identification hereafter.

COROLLARY 1. Let $\psi \in S$ such that $\psi = 1$ on a neighborhood of the origin, and let $w \in A_1$. Then a function f is in h_w^1 if and only $f \in L_w^1$ and $r_j^\psi f \in L_w^1$, $1 \le j \le n$, where $(r_j^\psi T)^\wedge = -(1-\psi)i(\xi_j/|\xi|)\hat{T}$ for $T \in S'$.

§3. Maximal function characterizations of H_w^1 and h_w^1

We begin by recalling the definition of H_w^p ([12]). Let $w \in A_\infty$ and $0 < p < \infty$. A function u, harmonic on the upper half space $R_+^{n+1} = R^n \times]0,\infty[$, is said to be in H_w^p if $N(u)(x) = \sup_{|x-y|<t}|u(y, t)| \in L_w^p$. The space H_w^p is equipped with the norm $||u||_{H_w^p} = ||N(u)||_{p,w}$. If $w \in A_1$, then it follows from [13] that $u \in H_w^1$ if and only if u is the Poisson integral (on the half space) of an L_w^1-function f whose Riesz transforms $R_1 f, \ldots, R_n f$ are in L_w^1. Here, by $R_j f$, $j = 1, \ldots, n$, we mean the pointwise limit

$$R_j f(x) = \lim_{\varepsilon\to 0} c_n \int_{|y|>\varepsilon} f(x-y)\frac{y_j}{|y|^{n+1}}dy, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.$$

Further, $||N(u)||_{1,w} \approx ||f||_{1,w} + \sum_1^n ||R_j f||_{1,w}$. By this reason, we identify H_w^1 with the space of all L_w^1-functions f with $R_j f \in L_w^1$, $j = 1, \ldots, n$.

THEOREM 2. Let $w \in A_\infty$, $0 < p < \infty$ and f be a measurable function with $\int |f(x)|(1+|x|)^{-n-1}dx < \infty$. Then the following statements are equivalent.

(A) $N^+(f)(x) = \sup_{0<t<\infty}|\psi_t * f(x)| \in L_w^p$ for some $\psi \in S$ with $\int \psi(x)dx = 1$.

(B) $N(f)(x) = \sup_{|x-y|<t}|\psi_t * f(y)| \in L_w^p$ for some ψ as above.

(C) $N^*(f)(x) = \sup_{\Phi \in \mathcal{B}_0} \sup_{|x-y|<t}|\Phi_t * f(y)| \in L_w^p$, where

$$\mathcal{B}_0 = \left\{\Phi \in \mathcal{S};\ \sum_{|\alpha| \le N_0,\ |\beta| \le N_0} \|x^\alpha D^\beta \Phi\|_\infty \le 1\right\}$$

and N_0 is a sufficiently large number depending on n, p and w.

(D) $N(u)(x) = \sup_{|x-y|<t} |u(y, t)| \in L^p_w$, where u is the Poisson integral of f on the upper half space, i. e., $u(x, t) = K_t * f(x)$ and $K_t(x) = c_n t(|x|^2 + t^2)^{-(n+1)/2}$.

Further, $\|N^+(f)\|_{p,w} \approx \|N(f)\|_{p,w} \approx \|N^*(f)\|_{p,w} \approx \|N(u)\|_{p,w}$.

The proof of Theorem 2 follows closely the paper [1] of Fefferman-Stein. We need nonetheless modifications. We also point out that the equivalence between $\|N^+(f)\|_{p,w}$ and $\|N(f)\|_{p,w}$ was not proved in [1]. The proof of (A) $\Rightarrow$ (B) of Theorem 11 in [1] depends on the finiteness of the L^p-norms of two auxiliary functions $u^*_{\varepsilon N}$ and $U^*_{\varepsilon N}$, and the inequality $\|U^*_{\varepsilon N}\|_p \le C\, \|u^*_{\varepsilon N}\|_p$. Since the constant C in this inequality depends on N which is a number depending on f, one would obtain the inequality $\|N(f)\|_p \le C\, \|N^+(f)\|_p$ with C depending on f. However, granted the finiteness of $\|N(f)\|_{p,w}$, it can be proved that C is independent of f as follows. Let $\beta > 1$, $x \in R^n$ and $\Gamma_\beta(x) = \{(z,s) \in R^{n+1}_+;\ |z-x| < \beta s\}$. Then there exists $c > 0$ such that $\{(z,t);\ |z-y| < ct\} \subset \Gamma_\beta(x)$ for all (y, t) with $|y-x| < t$. Fix such a (y, t). Let r be a number such that $0 < r < 1$, $0 < r < p$ and $w \in A_{p/r}$. Put $u = \psi_s * f$, $U = s|\nabla_z u|$ and $N_\beta(U)(x) = \sup_{\Gamma_\beta(x)} |U(z,s)|$. For $z \in B = \{z \in R^n;\ |z-y| < ct\}$, the mean value theorem gives

$$|u(y, t)|^r \le [N^+(u)(z)]^r + c^r[N_\beta(U)(x)]^r.$$

Noting that $B \subset \{z;\ |z-x| < (1+c)t\} = B'$, we derive from an integration over B that

$$|u(y, t)|^r \le \left(\frac{1+c}{c}\right)^n \frac{1}{|B'|}\int_{B'} N^+(u)^r dz + c^r[N_\beta(U)(x)]^r$$

$$\le \left(\frac{1+c}{c}\right)^n [N^+(u)^r]^*(x) + c^r[N_\beta(U)(x)]^r. \qquad (12)$$

On the other hand, the proof of (B) $\Rightarrow$ (C) implies that

$$||N_\beta(U)||_{p,w} \le C_1 \,||N(U)||_{p,w} \le C_2 \,||N(u)||_{p,w}.$$

This, combined with (12) and the weighted estimate for the Hardy maximal function [10], gives

$$||N(u)||_{p,w} \le C_3\left\{\left(\frac{1+c}{c}\right)^{n/r} ||N^+(u)||_{p,w} + c||N(u)||_{p,w}\right\}.$$

Since C_3 is independent of c, by choosing c small enough, we obtain the required estimate.

The above proof also serves as an example on how to deal with the weighted case.

<u>THEOREM 3</u>. Let $w \in A_\infty$, $0 < p < \infty$ and f be a measurable function with $\int |f(x)|(1+|x|)^{-n-1}dx < \infty$. Then the following statements are equivalent.

(A)' $N^+(f)(x) = \sup_{0<t<1}|\psi_t*f(x)| \in L^p_w$ for some $\psi \in S$ with $\int\psi(x)dx = 1$.

(B)' $N(f)(x) = \sup_{\Gamma(x)}|\psi_t*f(y)| \in L^p_w$ for some ψ as above.

(C)' $N^*(f)(x) = \sup_{\Phi \in B_1}\sup_{\Gamma(x)}|\Phi_t*f(y)| \in L^p_w$, where

$$B_1 = \left\{\Phi \in S;\ \sum_{|\alpha|\le N_1,\ |\beta|\le N_1} ||x^\alpha D^\beta \Phi||_\infty \le 1\right\}$$

and N_1 is a sufficiently large number depending on n, p and w.

(D)' Pf is in h^p_w.

Further, $||N^+(f)||_{p,w} \approx ||N(f)||_{p,w} \approx ||N^*(f)||_{p,w} \approx ||Pf||_{h^p_w}$.

Theorems 2 and 3 allow us to derive the connection between H^1_w

and h^1_w. Assume hereafter that $w \in A_1$ unless otherwise stated.

COROLLARY 2. (i) H^1_w is continuously embedded in h^1_w.

(ii) If ψ is a function in S such that $\int \psi(x)dx = 1$ and $\int x^\alpha \psi(x)dx = 0$ for all $|\alpha| \neq 0$, and $f \in h^1_w$, then $f - \psi * f \in H^1_w$ and $\| f - \psi * f \|_{H^1_w} \le C \| f \|_{h^1_w}$.

§4. Atomic decompositions

We first define atoms in our context. A function a, supported in a cube I, is called an (H^1_w, q)-atom $(1 < q \le \infty)$ if the following two conditions are satisfied:

(i) $\left\{\frac{1}{w(I)} \int_I |a(x)|^q w(x)dx\right\}^{1/q} \le w(I)^{-1}$;

(ii) $\int_I a(x)dx = 0$.

A function a, supported in a cube I, is called an (h^1_w, q)-atom $(1 < q \le \infty)$ if the condition (i) is satisfied and

(ii)' $\int_I a(x)dx = 0$ if $|I| < 1$.

The atomic decomposition was first studied for $H^p(R^1)$ by Coifman ([15]) and was later extended to $H^p(R^n)$ by Latter ([16]). The study in the context of space of homogeneous type is given in [17] with many extensions and applications of Hardy spaces.

THEOREM 4. A function f is in H^1_w (resp. h^1_w) if and only if there exist a sequence of (H^1_w, ∞)-atoms (resp. (h^1_w, ∞)-atoms) $\{a_j\}$ and a sequence $\{\lambda_j\}$ such that

$$f = \Sigma \lambda_j a_j \text{ in } L^1_w \text{ and } \Sigma |\lambda_j| < \infty.$$

Further,

$$\|f\|_{H^1_w} \approx \inf\left\{\Sigma|\lambda_j|;\ f = \Sigma\lambda_j a_j \text{ in } L^1_w,\ a_j\text{: } (H^1_w,\infty)\text{-atoms}\right\}$$

and

$$\|f\|_{h^1_w} \approx \inf\left\{\Sigma|\lambda_j|;\ f = \Sigma\lambda_j a_j \text{ in } L^1_w,\ a_j\text{: } (h^1_w,\infty)\text{-atoms}\right\}.$$

It is obvious that (H^1_w,∞)-atoms (resp. (h^1_w,∞)-atoms) are (H^1_w,q)-atoms (resp. (h^1_w,q)-atoms). A partial converse is true.

PROPOSITION. Let $w \in A_q$ $(1 < q < \infty)$ and a be an (H^1_w,q)-atom (resp. (h^1_w,q)-atom). Then there exist a sequence of (H^1_w,∞)-atoms (resp. (h^1_w,∞)-atoms) $\{a_j\}$ and a sequence $\{\lambda_j\}$ such that

$$a = \Sigma\lambda_j a_j \text{ in } L^1_w \text{ and } \Sigma|\lambda_j| \le C,$$

where C is a constant not depending on the given atom a.

The atomic decompositions provide us an easy way to describe dual spaces. Since the dual of H^1_w was characterized earlier by Muckenhoupt-Wheeden by the Fefferman-Stein method ([18]) and by Garcia-Cuerva by the atomic decomposition method ([19]), we give only the result for h^1_w. We need one more definition. A locally integrable function b is said to be in bmo_w if $\|b\|^{**}_w = \max\left\{\sup_{|I|<1} (1/w(I))\int_I |b(x)-b_I|dx,\ \sup_{|I|\ge 1}(1/w(I))\int_I |b(x)|dx\right\} < \infty$.

COROLLARY 3. (i) The dual of h^1_w is bmo_w in the sense that, if L is a continuous linear functional on h^1_w, then there exists a unique $b \in bmo_w$ with the property that

$$Lf = \int b(x)f(x)dx$$

for any f which is a linear combination of (h^1_w,∞)-atoms; and conversely, any such L, initially defined for linear combinations of (h_w^1,∞)-atoms, can be uniquely extended to a continuous linear functional on h_w^1. Further, $\|L\| \approx \|b\|_w^{**}$.

(ii) A locally integrable function b is in bmo_w if and only if $\max\left\{\sup_{|I|<1}(1/w(I))\int_I |b(x)-b_I|^q w(x)^{1-q}dx,\ \sup_{|I|\geq 1}(1/w(I))\int_I |b(x)|^q w(x)^{1-q}dx\right\} < \infty$, $1 \leq q < \infty$.

Proofs of the results in sections 2-4 and related matters are given in [20] and will be appeared elsewhere.

§5. Remarks and comments

Our first remark concerns the results of Garcia-Cuerva. In the paper [19], Garcia-Cuerva studied the space $H_w^p(R_+^2) = H_w^p$ of analytic functions on the upper half plane, where $0 < p < \infty$ and $w \in A_\infty$. He has obtained, among other things, the atomic decomposition for any $f \in L_w^q$ whose Poisson integral is the real part of a function in H_w^p; here $q > q_0 = \inf\{s;\ w \in A_s\}$ and $0 < p \leq 1$. Hence, although there is an overlap, his results in case $w \in A_1$ and $p = 1$ do not entirely contain ours for $H_w^1(R_+^2)$. The main difficulty in the case $w \in A_\infty$ is that we are not able to prove the existence of boundary values in the sense of distributions of functions in H_w^p and also to find a way to explicitly express these boundary values in terms of the corresponding harmonic functions in H_w^p.

Another possible approach to weighted Hardy spaces, bypassing harmonic functions or analytic functions, is to study the space of all tempered distributions f whose non-tangential maximal functions $N(f) \in L_w^p$, where the weight function w may not be in the class A_∞. At least, if $w \in A_\infty$, then equivalence between (A), (B) and (C) (resp.

(A)', (B)' and (C)') in Theorem 2 (resp. Theorem 3) holds for an arbitrary $f \in S'$.

Our final remark is about the Triebel-Lizorkin spaces. The identities (1) and (2) in section 1 suggest the possibility of characterizing $F^s_{p,q}$ and $\dot{F}^s_{p,q}$ via traces of solutions of differential equations. This type of characterization is well-known for Besov spaces ([3], [21], [22], [23]). Another subject that should be worth studying is weighted Besov spaces and weighted Triebel-Lizorkin spaces and their relations with weighted Hardy spaces. The preparation for such a study has been given by Triebel ([24]).

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