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A note on the stable $\mathbb{Z}_2$ - cohomotopy groups

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§ 0. Introduction

Let X and Y be based $\mathbb{Z}_2$ - complexes and let $\tilde{\pi}^{n,m}(X ; Y)$ denote the group of stable $\mathbb{Z}_2$ - maps of degree (n, m) (see §1 for the definition). If $Y = \Sigma^{0,0}$ is the o - sphere, then $\tilde{\pi}^{n,m}(X : \Sigma^{0,0}) = \tilde{\pi}^{n,m}(X)$ is called the stable $\mathbb{Z}_2$ - cohomotopy group of X, and has been studied by various authors ([3], [4], [7] and [9]). The purpose of this note is to describe it in terms of non equivariant stable homotopy for certain X and Y.

To state our result recall that the stunted projective space P_a^b is defined for all integers a and b as a stable complex by the well known periodicity. We shall define in §2 a stable map $u : \Sigma^{-1} \longrightarrow P_a^b$ for $b \geq -1$ and a well defined stable homotopy type P_a^b / Σ^{-1}. Let S^q denote the q - sphere with the antipodal involution. Then our results are,

Theorem 1. Let q be a positive integer and let X be a based $\mathbb{Z}_2$ - complex with the trivial $\mathbb{Z}_2$ - action. Then for any $n, m \in \mathbb{Z}$, there is a natural stable isomorphism

$$\bar{\alpha} : \tilde{\pi}^{n,m}(X : S^q_+) \cong \tilde{\pi}^m(X : P^{n+q}_n).$$

<u>Theorem 2.</u> Let n,m and q be integers such that $n + q > 0$. Then for any based $\mathbb{Z}_2$ - complex X with the trivial $\mathbb{Z}_2$ - action of dim. $< n + m + q$, there is a natural stable isomorphism

$$\alpha : \tilde{\pi}^{n,m}(X) \cong \tilde{\pi}^m(X : P^{n+q}_n / \Sigma^{-1}).$$

Here $\tilde{\pi}^m(\; : \;)$ denotes the usual group of (non - equivariant) stable maps of degree m. If we fix m, then the above groups are all generalized cohomology theoryes and by <u>stable</u> we mean that those isomorphisms commute with the suspension isomorphism. We should mention about the dimensional restriction in Theorem 2. If we use a homotopy type P^∞_n / Σ^{-1} (defined in a non - canonical way), we can state that there is an isomorphism (not natural !)

$$\tilde{\pi}^{n,m}(X) \cong \tilde{\pi}^m(X : P^\infty_n / \Sigma^{-1})$$

for any <u>finite</u> trivial $\mathbb{Z}_2$ - complex X.

Let $n = m = 0$, then $P^\infty_0 / \Sigma^{-1} \simeq RP^\infty_+ \vee \Sigma^0$. In this case our result is just the theorem of Segal [7]. When $n > 0$ and X is a sphere similar results are obtained by [7].

Finally we state a conjecture which is seen to be equivalent to the conjecture of Mahowald [1] by using Theorem 1 and 2. Let $\pi_{n,m} = \tilde{\pi}^{-n,-m}(\Sigma^{0,0})$ be the stable (n, m) - stem. Using the inclusion i $\Sigma^{p,q} \longrightarrow \Sigma^{p+1,q}$, one can define an inverse system $\{\pi_{n,m}\}_n$.

<u>Conjecture.</u> $\varprojlim_n \pi_{n,m} = 0$ for all m.

§ 1. Some notations

First we recall some notations. If X is a $\mathbb{Z}_2$ - space, $X^{\mathbb{Z}_2}$ dinotes the fixed point subspace. $\mathbb{R}^{n,m}$ denotes the representation of $\mathbb{Z}_2$ on $\mathbb{R}^{n+m}$ given by

$\tau(x_1,\ldots, x_n, x_{n+1}\ldots x_{n+m}) = (-x_1,\ldots,-x_n, x_{n+1},\ldots,x_{n+m})$. $\Sigma^{n,m}$ denotes the one point compactification of $R^{n,m}$. The unit sphere in $R^{r+1,0}$ is a free $\mathbb{Z}_2$ - complex and denoted by S^r. Let X be a based $\mathbb{Z}_2$ - space, then $\Omega^{n,m}X$ denotes the function space $\mathrm{Map}(\Sigma^{n,m}, * : X, *)$ with the compact open topology and the usual $\mathbb{Z}_2$ - action.

The equivariant infinite loop space of X is defined by $Q_{\mathbb{Z}_2}(X) = \varinjlim \Omega^{n,m}(\Sigma^{n,m}X)$ similarly to the non - equivariant case. It is known [2] that if X has a $\mathbb{Z}_2$ - homotopy type of a $\mathbb{Z}_2$ - complex, then so does $\Omega^{n,m}X$ (and hence $Q_{\mathbb{Z}_2}(X)$).

A $\mathbb{Z}_2$ - spectrum $\mathbb{X} = \{X_n, \mathcal{E}_n\}$ is defined [3] by $\mathbb{Z}_2$ - spaces X_n and structure maps $\mathcal{E}_n : \Sigma^{1,1}X_n \longrightarrow X_{n+1}$. Given a $\mathbb{Z}_2$ - complex X the suspension spectrum (with a shifted dimension) $\Sigma^k X$ is defined by $(\Sigma^k X)n = \Sigma^{n+k,n+k}X$ where $k \leftarrow \mathbb{Z}$, and is reffered to a stable complex and sometimes written as $\Sigma^k X$. A $\mathbb{Z}_2$ - spectra map (or stable

$\mathbb{Z}_2$ - map) is defined similarly as the non - equivariant case.

Let X and Y be stable complexes. Then the group $\tilde{\pi}^{n,m}(X : Y)$ of stable homotopy classes of stable maps of degree (n, m) is defined by

$$\tilde{\pi}^{n,m}(X : Y) = \varinjlim_p [\Sigma^{p,p}X, \Sigma^{n+p,m+p}Y]_{\mathbb{Z}_2}.$$

If (n, m) = (0, 0) it is sometimes denoted by $\{X, Y\}_{\mathbb{Z}_2}$.

$\tilde{\pi}^{n,m}(X : \Sigma^{0,0})$ is simply denoted by $\tilde{\pi}^{n,m}(X)$ and called the (n, m) - dim. Stable $\mathbb{Z}_2$ - cohomotopy group.

Given a $\mathbb{Z}_2$ - spectrum $\mathbb{X}$, we can define the associated $\mathbb{Z}_2$ - Ω - spectrum $Q\mathbb{X}$ by $(Q\mathbb{X})n = \varinjlim_q \Omega^{q,q}Xn{+}q$. Note that $((Q\mathbb{X})n)^{\mathbb{Z}_2} = \varinjlim_q \Omega^q(\Omega^{q,o}Xn{+}q)^{\mathbb{Z}_2}$ is an infinite loop space.

Hence the fixed point spectrum $(Q\mathbb{X})^{\mathbb{Z}_2}$ is an Ω- spectrum which we denote by $\mathbb{E}(\mathbb{X})$. Eviclently we see that $\mathbb{E}(\Sigma^{o,n}\mathbb{X}) \cong \Sigma^n\mathbb{E}(\mathbb{X})$.

The infinite loop space $\mathbb{E}(\mathbb{X})o$ is denoted by $E(\mathbb{X})$.

Clearly $\mathbb{E}$ and E are functor. We remark that $\mathbb{E}(\mathbb{X})$ is not equivalent to the fixed point subspectrum $\mathbb{X}^{\mathbb{Z}_2}$. Given a $\mathbb{Z}_2$ - complex X and n, m $\leftarrow \mathbb{Z}$, consider the stable complex $\Sigma^{n,m}X$ (i.e., X with dimension shifted). If n and m are positive, then $E(\Sigma^{n,m}X)$ $= Q_{\mathbb{Z}_2}(\Sigma^{n,m}X)^{\mathbb{Z}_2}$. Therefore for negative n or m, we often write as $Q_{\mathbb{Z}_2}(\Sigma^{n,m}X)^{\mathbb{Z}_2}$ instead of $E(\Sigma^{n,m}X)$.

Under the above notations the following lemmas are obvious.

Lemma 3. Let Y be a $\mathbb{Z}_2$ - complex and X a $\mathbb{Z}_2$ - complex with the trivial $\mathbb{Z}_2$ - action. Then there is a natural isomorphism.

$$\tilde{\pi}^{n,m}(X : Y) \cong [X, E(\Sigma^{n,m}Y)].$$

Lemma 4. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a $\mathbb{Z}_2$ - cofibration (see [3] for definition) of stable $\mathbb{Z}_2$ - complexes. Then

$$\mathbb{E}(X) \xrightarrow{\mathbb{E}(f)} \mathbb{E}(Y) \xrightarrow{\mathbb{E}(g)} \mathbb{E}(Z)$$

is a cofibration of spectra. Therefore the sequence

$$E(X) \xrightarrow{E(f)} E(Y) \xrightarrow{E(g)} E(Z)$$

is homotopy equivalent to a fibration.

§ 2. Stunted projective spaces.

Let ξ be the canonical line bundle over the projective space RP^a. There is a canonical isomorphism

$$KO(RP^a) \cong KO_{Z_2}(S^a)$$

induced from the projection $S^a \longrightarrow RP^a$ (see [4]).

By this isomorphism ξ corresponds to the Z_2 - vector bundle $S^a \times R^{1,0}$.

Therefore the space $S^a_+ \wedge_{Z_2} \Sigma^{n,o} = (S^a_+ \wedge \Sigma^{n,o}) / Z_2$ is identified with the thom complex $T(n\xi)$. It is well known [4] that $T(n\xi)$ is homeomorphic to the stunted projective space $P^{n+a}_n = RP^{n+a} / RP^{n-1}$.

It is well known that the order of $\xi - 1 \in \widetilde{KO}(RP^a)$ is finite.

Then the following lemma is obvious.

Lemma 5. Let d be a multiple of the order of $\xi - 1 \in KO(RP^a)$. Then there is a Z_2 - vector bundle isomorphism

$$\eta : S^a \times R^{d,o} \longrightarrow S^a \times R^{o,d}.$$

We denote by the same η the induced Z_2 - homeomorphism $S^a_+ \wedge \Sigma^{d,o} \longrightarrow S^a_+ \wedge \Sigma^{o,d}$, and also the homeomorphism $P^{a+d}_a \longrightarrow \Sigma^d P^a_o$. Now for each $a \geq 0$ and $n \in Z$ choose d as above satisfying $d+n \geq 0$, and define a stable homotopy type P_n^{n+a} by $\Sigma^{-d}P_{n+d}^{n+d+a}$

Then by the percodicity η, P_n^{n+a} is well defined. For the stable complex P_n^{n+a}, define an infinite loop space $Q(P_n^{n+a})$ by $\Omega^d Q(P_{n+d}^{n+d+a})$.

Next we define a stable homotopy type P_n^{n+a}/Σ^{-1} for $n + a \geq -1$.

First let $n \geq 0$, then we define P_n^{n+a}/Σ^{-1} to be $P_n^{n+a} \vee \Sigma^o$, namely the cofibre of the unigue homotopy class $\Sigma^{-1} \longrightarrow P_n^{n+a}$, where Σ^m denotes the m - sphere spectrum. Next let $n < 0$ and put $r = -n > 0$. Let d be a multiple of the order of $\xi - 1 \in \widetilde{KO}(RP^a)$, and choose a periodcity

$$\eta : S^a \times R^{d,o} \cong S^a \times R^{o,d}.$$

Clearly it restricts to a periodicity $\eta : S^{r-1} \times R^{d,o} \cong S^{r-1} \times R^{o,d}$.

Note that $a > r$. Let ν be the normal bundle of an embedding $RP^{r-1} \subset R^N$, N large enough. It is well known that such embeddings are isotopic to each other. Since we have canonical isomorphisms $\tau(RP^{r-1}) \oplus \varepsilon^1 \cong r\xi$ and $\tau(RP^{r-1}) \oplus \nu \cong RP^{r-1} \times R^N$, we have a canonical isomorphism

$$d\xi \oplus \nu \cong (d - r)\xi \oplus (N + 1)\varepsilon^1.$$

Then by using $\eta : d\xi \cong d\varepsilon^1$, we have a bundle isomorphism $\varphi : \nu \oplus d\varepsilon^1 \cong (d - r)\xi \oplus (N + 1)\varepsilon^1$. Let $h : \Sigma^N \longrightarrow T(\nu)$ be the Pontrjagin -

Thom map of the embedding $\nu \subset R^{N}$. By the unigueness of normal bundles up to isotopy the homotopy class of h is uniquely defermined. Then define a stable map $u : \Sigma^{-1} \longrightarrow P_{-r}^{-r+a} = P_{n}^{n+a}$ by the composite

$$\Sigma^{N+d} \xrightarrow{h} \Sigma^{d} \wedge T(\nu) \overset{\varphi}{=} \Sigma^{N+1} T((d-r)\xi) = \Sigma^{N+1} P_{d-r}^{d-1} \xrightarrow{i} \Sigma^{N+1} P_{d-r}^{d-r+a}.$$

If we change a periodicity by $\eta' : S^{r+a} \times R^{d,o} \longrightarrow S^{r+a} \times R^{o,d}$, then the resulting map u' differs from u by a self homotopy equivalence of P_{d-r}^{d-r+a}. In fact note that ν is a restriction of a vector bundle over RP^{a}.

Then bundle isomorphisms $\mu \oplus id$, $\mu' \oplus id : d\xi \oplus \nu \longrightarrow d\varepsilon^{1} \oplus \nu$ extend to bundle isomorphisms over RP^{a}, and hence $(\mu \oplus id) \circ (\mu' \oplus id)^{-1}$ is a bundle automorphism of $(d-r)\xi \oplus (N+1)\varepsilon^{1}$ over RP^{a}. This shows the vequired property.

Now let P_{n}^{n+a}/Σ^{-1} be the cofibre of u. The above argument shows that the stable homotopy type of P_{n}^{n+a}/Σ^{-1} does not depend on choices of d and η. It is obvious that the n + b skeleton $(P_{n}^{n+a}/\Sigma^{-1})^{(n+b)}$ is homotopy equivalent to P_{n}^{n+b}/Σ^{-1}. Therefore we can define (not canonically) a stable homotopy type $P_{n}^{\infty}/\Sigma^{-1}$.

It will be usefull to give another description of u.

Let $S^{r-1} \subset S^{r-1} \times \Sigma^{d-r,0}$ be the obvious embedding. The normal bundle is then canonically isomorphic to $S^{r-1} \times R^{d-r,0}$.

Hence the normal bundle $\nu(RP^{r-1}, S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0})$ of the induced embedding $RP^{r-1} \subset S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0}$ is identified with $(d - r)\xi$.

We remark that $\tau(S^{r-1} \times \Sigma^{d-r,0}) \oplus \varepsilon = S^{r-1} \times \Sigma^{d-r,0} \times R^{d,0}$, and hence by the periodicity μ we have an isomorphism

$$\tau(S^{r-1} \times_{\mathbb{Z}_2} {}^{d-r,0}) \oplus \varepsilon \cong d\varepsilon .$$

Thus $S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0}$ is a framed manifold and μ gives a framing.

Then given an embedding $S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0} \subset R^{N}$, we have the Pontrjagin - Thom map

$$\Sigma^{N} \longrightarrow \Sigma^{N-d+1}(S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0}{}_{+}).$$

It is easy to see that the map u defined above is also given by the composite

$$\Sigma^{N} \longrightarrow \Sigma^{N-d+1}(S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0}{}_{+}) \longrightarrow \Sigma^{N-d+1}(T((d - r)\xi))$$
$$= \Sigma^{N-d+1} P_{d-r}^{d-1} \hookrightarrow \Sigma^{N-d+1} P_{d-r}^{d-r+a} = \Sigma^{N+1} P_{n}^{n+a}$$

where the second map is the Pontrjagin - Thom map of the embedding $RP^{r-1} \longrightarrow S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0}$.

§ 3. The space $Q_{\mathbb{Z}_2}(X)^{\mathbb{Z}_2}$

Let X be a finite $\mathbb{Z}_2$ - complex and X_+ denotes X with the disjoint base point. Given a continious map $f : \Sigma^m \longrightarrow X_+^{\mathbb{Z}_2} \wedge \Sigma^m$, let $e(f)$ be the composite

$$\Sigma^{n,m} \xrightarrow{\Sigma^{n,o}f} X_+^{\mathbb{Z}_2} \wedge \Sigma^{n,m} \subset X_+ \wedge \Sigma^{n,m}$$

This defines a continious map $e : Q(X_+^{\mathbb{Z}_2}) \longrightarrow Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2}$.

Conversely by assigning to a $\mathbb{Z}_2$ - map $f : \Sigma^{n,m} \longrightarrow X_+ \wedge \Sigma^{n,m}$ the map $f^{\mathbb{Z}_2}$ of fixed point sets, we obtain a continious map

$$\varphi : Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2} \longrightarrow Q(X_+^{\mathbb{Z}_2}).$$

It is obvious that $\varphi \circ e = \text{id}$. More precisely we have

Proposition 6. There is a natural homotopy equivalence

$$\lambda : Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2} \longrightarrow Q(X_+^{\mathbb{Z}_2}) \times Q((S^\infty \times_{\mathbb{Z}_2} X)_+).$$

Moreover via λ, the maps e and φ are homotopic to the canonical inclusion and projection, respectively.

Proof. The existence of λ is shown in [8]. In [8], λ is defined by a geometric method. That is, we may suppose that X is a G - manifold and let Y be a manifold. Then any element of $[Y_+, Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2}]$ is represented by a pair

$$X \xleftarrow{f} E \xrightarrow{h} Y$$

where E is a G - manifold, f is a G - map and h is a framed map. (see [γ] for definition).

Then it is known that E is decomposed into a disjoint sum of submanifolds $E_0 \amalg E_1$, where E_0 is trivial and E_1 is free as $\mathbb{Z}_2$ - space.

Then the map λ is induced from this decomposition.

Then checking for a geometric representative, we see that the homomorphism

$$e_* : [Y_+, Q(X_+^{\mathbb{Z}_2})] \longrightarrow [Y_+, Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2}]$$

coincides with the canonical inclusion. For the map φ the proof is similar.

Now we shall stabilize the above result. Let n and m be positive integers. We have a $\mathbb{Z}_2$ - cofibration

$$X_+ \xrightarrow{i} (X \times \Sigma^{n,m})_+ \xrightarrow{\pi} X_+ \wedge \Sigma^{n,m}$$

and the projection $p : (X \times \Sigma^{n,m})_+ \longrightarrow X_+$ such that $poi = id_X$.

Applying the functor $Q_{\mathbb{Z}_2}(\ )^{\mathbb{Z}_2} = E(\ )$, we obtain a fibration (up to equivalence)

$$E(X_+) \xrightarrow{E(i)} E((X \times \Sigma^{n,m})_+) \xrightarrow{E(\pi)} E(X_+ \wedge \Sigma^{n,m}).$$

It is obvious that the fibration is trivial, and by using the map E(p) we can define a canonical splitting

$$s : E(X_+ \wedge \Sigma^{n,m}) \longrightarrow E((X \times \Sigma^{n,m})_+).$$

Since the homotopy equivalence of Proposition 6 is natural, we easily see the following generalization of Proposition 6.

Lemma 7. Let n and m be positive integers, then there exists a natural homotopy equivalence

$$\lambda : E(X_+ \wedge \Sigma^{n,m}) \longrightarrow Q(X_+^{\mathbb{Z}_2} \wedge \Sigma^{m}) \times Q((X \times S^\infty)_+ \wedge_{\mathbb{Z}_2} \Sigma^{n,m}).$$

Lemma 8. The map λ is an infinite loop map.

Proof. We are enough to show that the following diagram is commutative.

$$\begin{array}{ccc}
[Y_+, E(X_+)] & \xrightarrow{\lambda_*} & [Y_+, Q(X_+^{\mathbb{Z}_2})] \oplus [Y_+, Q((X \times_{\mathbb{Z}_2} S^\infty)_+)] \\
\sigma \downarrow \cong & & \cong \downarrow \sigma \\
[Y_+ \wedge \Sigma^{o,n}, E(X_+ \wedge \Sigma^{o,n})] & \xrightarrow{\lambda_*} & [Y_+ \wedge \Sigma^{n}, Q(X_+^{\mathbb{Z}_2} \wedge \Sigma^{n})] \oplus [Y_+ \wedge \Sigma^{n}, Q((X \times S^\infty)_+ \wedge_{\mathbb{Z}_2} \Sigma^{o,n})],
\end{array}$$

where σ is the suspension isomorphism. We may suppose that X and Y are manifolds as before, and we can take a pair $x = (X \xleftarrow{f} E \xrightarrow{h} Y)$ for a representative of an element of $[Y_+, E(X_+)]$. Note that $h \times id : E \times \Sigma^{o,n}$ is canonically framed. Hence define a homomorphism

$$\bar{\sigma} : [Y_+, E(X_+)] \longrightarrow [(Y \times \Sigma^{o,n})_+, E((X \times \Sigma^{o,n})_+)]$$

by $\bar{\sigma}(x) = (X \times \Sigma^{o,n} \xleftarrow{f \times id} E \times \Sigma^{o,n} \xrightarrow{h \times id} Y \times \Sigma^{o,n})$. Then we easily see that the diagram

$$\begin{array}{ccc} & \overset{\sigma}{\nearrow} & [Y_+ \wedge \Sigma^{o,n}, E(X_+ \wedge \Sigma^{o,n})] \\ [Y_+, E(X_+)] & & \downarrow j \\ & \underset{\bar{\sigma}}{\searrow} & [(Y \times \Sigma^{o,n})_+, E((X \times \Sigma^{o,n})_+)] \end{array}$$

is commutative, where j is the split monomorphism induced from the map s. Then the commutativity of the rectangle diagram is shown easily for a geometric representative $(X \xleftarrow{f} E \xrightarrow{h} Y)$. q. e. d.

Given a CW - complex X, let ΣX denotes the suspension spectrum. Then from the above lemmas, we obtain a homotopy equivalence of spectra

$$\bar{\lambda} : E(X_+ \wedge \Sigma^{n,m}) \simeq \Sigma(X_+^{\mathbb{Z}_2} \wedge \Sigma^m) \vee \Sigma((X \times S^\infty)_+ \underset{\mathbb{Z}_2}{\wedge} \Sigma^{n,m}).$$

Let $n(>0)$ and m be integers. Then

<u>Proposition 9.</u> There exists an equivalence of spectra

$$\bar{\lambda} : E(X_+ \wedge \Sigma^{n,m}) \simeq \Sigma(X_+^{\mathbb{Z}_2} \wedge \Sigma^m) \vee \Sigma((X \times S^\infty)_+ \wedge_{\mathbb{Z}_2} \Sigma^{n,m})$$

§ 4. Proof of the theorems

In this section we proves Theorem 1 and Theorem 2 simultaniously.

By Lemma 3 we have natural isomorphisms

$$\tilde{\pi}^{n,m}(X : S^q_+) \cong [X, \mathbb{E}(\Sigma^{n,m} S^q_+)]$$

and

$$\tilde{\pi}^{n,m}(X) \cong [X, \mathbb{E}(\Sigma^{n,m})].$$

Thus the problem is to determine those spectra $\mathbb{E}(\Sigma^{n,m})$ and $\mathbb{E}(\Sigma^{n,m} S^q_+)$.

for any $n, m \in \mathbb{Z}$.

First we suppose that $n \geq 0$. Note that $(S^q)^{\mathbb{Z}_2} = \phi$ and $S^q \times S^\infty$

is $\mathbb{Z}_2$ - homotopy equivalent to S^q. Then by Lemma 7 and Lemma 8,

we easily see that

$$\mathbb{E}(\Sigma^{n,m} S^q_+) \simeq \Sigma(\Sigma^{-m} P_n^{n+q})$$

and

$$\mathbb{E}(\Sigma^{n,m}) \simeq \Sigma(\Sigma^{m}) \vee \Sigma(\Sigma^{-m} P_n^{\infty})$$

as spectra for any $m \in Z$. This immedeately implies the theorems for

$n \geq 0$.

Next suppose that $n < 0$, and put $r = -n > 0$.

Given a positive integer q, let d be an integer $(d \geq r)$ and

$\mu : \Sigma^{o,d} S^q_+ \longrightarrow \Sigma^{d,o} S^q_+$ be a periodicity as in Lemma 5. Let N, M be integers large enough. Given a $\mathbb{Z}_2$ - map $f : \Sigma^{N+r,M} \longrightarrow \Sigma^{N,M+m} S^q_+$, let $\mu^*(f)$ be the composite

$$\Sigma^{N+r,M} \xrightarrow{f} \Sigma^{N,\, M+m} S^q_+ \overset{\mu}{\cong} \Sigma^{N+d,M-d+m} S^q_+ .$$

Then we obrain an isomorphism (periodicity) of spectra.

$$\mu * : \mathbb{E}(\Sigma^{n,m} S^q_+) \longrightarrow \mathbb{E}(\Sigma^{n+d,m-d} S^q_+).$$

Since $n + d \geq 0$, we can reduce to the first case and we have

$$\mathbb{E}(\Sigma^{n+d,m-d} S^q_+) \simeq \Sigma(\Sigma^{m-d} P^{n+d+q}_{n+d}) = \Sigma(\Sigma^m P^{n+q}_n).$$

This shows Theorem 1 for $n < 0$.

Next let n, m and q be as above, and suppose that $n + q > 0$. From the standard $\mathbb{Z}_2$ - cofibration $S^q_+ \xrightarrow{P} \Sigma^{o,o} \xrightarrow{i} \Sigma^{q+1,o} \xrightarrow{\pi} \Sigma^{o,1} S^q_+$, where P is the unique non - trivial map, i is the standard inclusion and $\pi : \Sigma^{q+1,0} \longrightarrow \Sigma^{q+1,0}/\Sigma^{0,0} \cong \Sigma^{0,1} S^q_+$ is the projection, we obtain a stable $\mathbb{Z}_2$ - cofibration

$$\Sigma^{n,m} S^q_+ \xrightarrow{p} \Sigma^{n,m} \xrightarrow{i} \Sigma^{n+q+1,m} \xrightarrow{c} \Sigma^{n,m+1} S^q_+ .$$

Then by Lemma 4 we obtain a stable cofibration

$$\mathbb{E}(\Sigma^{n+q+1,m-1}) \xrightarrow{\mathbb{E}(c)} \mathbb{E}(\Sigma^{n,m} S^q_+) \xrightarrow{\mathbb{E}(p)} \mathbb{E}(\Sigma^{n,m}).$$

Choose d such as $n + d \geq 0$ and a periodicity

$$\mu * : \mathbb{E}(\Sigma^{n,m}S^q_+) \cong \mathbb{E}(\Sigma^{n+d,m-d}S^q_+).$$

Recall that $\mathbb{E}(\Sigma^{n+d,m-d}S^q_+) = \Sigma(\Sigma^{m-d}P^{n+d+q}_{n+d}) = \Sigma(\Sigma^m P^{n+q}_n)$.

By assumption $n + q + 1 > 0$ and by Lemma 8 we have an equivalence of spectra

$$\bar{\lambda} : \mathbb{E}(\Sigma^{n+q+1,m-1}) \simeq \Sigma(\Sigma^{m-1}) \vee \Sigma(\Sigma^{m-1}P^{\infty}_{n+q+1})$$

Using the equivalences $\mu *$ and $\bar{\lambda}$, the map $\mathbb{E}(c)$ is homotopic to a map $w : \Sigma(\Sigma^{m-1}) \vee \Sigma(\Sigma^{m-1}P^{\infty}_{n+q+1}) \longrightarrow \Sigma(\Sigma^m P^{n+q}_n)$. Let

$$u' : \Sigma(\Sigma^{m-1}) \longrightarrow \Sigma(\Sigma^m P^{n+q}_n)$$

be the restriction of w to $\Sigma(\Sigma^{m-1})$. Note that $\Sigma(\Sigma^{m-1}P^{\infty}_{n+q+1})$ is $n + m + q$ - connected. Hence the cofibre of u' is $n + m + q$ - equivalent to $\mathbb{E}(\Sigma^{n,m})$.

Then Theorem 2 for $n < 0$ follows immedeately from the following

Lemma 9 The cofibre of u' is stably homotopy equivalent to $\Sigma^m(P^{n+q}_n/\Sigma^{-1})$ of § 2.

Proof. We may suppose that $m = 0$. Since we have defined u' using a periodicity $\mu : S^q \times R^{d,o} \longrightarrow S^q \times R^{o,d}$, it is enough to show that $u' : \Sigma(\Sigma^{-1}) \longrightarrow \Sigma(P^{n+q}_n)$ is homotopic to u for the same choice of μ.

Let

$$\beta = (u')_* : \tilde{\pi}^{-1}(Y_+) \longrightarrow \tilde{\pi}^{o}(Y_+,\ P_n^{n+q})$$

be the induced homomorphism. Let $r = -n > 0$. and let

$z \in \pi^{-r,1}(\Sigma^{o,o}\ ;\ S^q_+)$ be the class of the composite

$$\Sigma^{r,o} \xrightarrow{\pi} \Sigma^{0,1} S^{r-1}_+ \subset \Sigma^{0,1} S^q_+.$$

By the smash product we have a pairing

$$\wedge : \tilde{\pi}^{a,b}(X : Y) \otimes \tilde{\pi}^{a',b'}(X' : Y') \longrightarrow \tilde{\pi}^{a+a',b+b'}(X \wedge X' : Y \wedge Y').$$

Let γ be the composite

$$\tilde{\pi}^{-r,1}(\Sigma^{o,o} : S^q_+) \otimes \tilde{\pi}^{o}(Y_+) \xrightarrow{id \otimes e_*} \tilde{\pi}^{-r,1}(\Sigma^{o,o} : S^q_+) \otimes \tilde{\pi}^{o,o}(Y_+)$$

$$\xrightarrow{\wedge} \tilde{\pi}^{-r,1}(Y_+, S^q_+) \underset{\cong}{\xrightarrow{\mu_*}} \tilde{\pi}^{d-r,1-d}(Y_+, S^q_+) \xrightarrow{\lambda_*} \tilde{\pi}(Y_+, P_n^{n+q}).$$

Then we see that $\beta(x) = \gamma(z \otimes x)$, and we are enough to show that $\gamma(Z \otimes 1) = \{u\}$. We remark that the element Z is given by the Pontrjagin - Thom construction of the pair $(* \longleftarrow S^{r-1} \hookrightarrow S^q)$, where the unique map $S^r \longrightarrow *$ is, say, $(r, -1)$ - framed.

That is, $\tau(S^{r-1}) \oplus \varepsilon = S^{r-1} \times R^{r,o}$. Then as in § 2, $S^{r-1} \times_{Z_2} \Sigma^{d-r,o}$ is a framed manifold by e of the periodicity μ. Then by the definition of λ we easily see that the class $\gamma(z \otimes 1)$ is given by the composite

$$\Sigma^N \xrightarrow{c} \Sigma^{N-d+1}((S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,o})_+) \xrightarrow{i} \Sigma^{N-d+1}((S^q \times_{\mathbb{Z}_2} \Sigma^{d-r,o})_+)$$

$$\xrightarrow{\pi} \Sigma^{N-d+1}(S^q_+ \wedge_{\mathbb{Z}_2} \Sigma^{d-r,o}) = \Sigma^{N+1}(P_n^{n+q})$$

where c is the Pontrjagin - Thom map, and i and π are obvious maps.

Now by the second description of the map u, we easily see that

$\{u'\} = r(Z\otimes 1) = \{u\}$.

This completes the proof.

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