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SOME REMARKS ON REGULAR * SEMIGROUPS

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A semigroup S is called to be <u>fundamental</u> if its only one congruence contained in the Green's relation $\mathcal{H}$ on S is the trivial one. In his paper [2], Hall gives us the construction of a fundamental regular semigroup which is the generalization of [1] and [3]. In this paper, we shall study a fundamental regular * semigroup.

A semigroup S with a unary operation $*: S \to S$ is called a <u>* semigroup</u> if it satisfies

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^*x^*$.

Let S and T be * semigroups. A mapping $\phi: S \to T$ is called a <u>* homomorphism</u> if ϕ is a (semigroup) homomorphism and $x^*\phi = (x\phi)^*$ for all x in S. A relation ν on S is called a <u>* relation</u> on S if $(x,y) \in \nu$ implies $(x^*,y^*) \in \nu$. A * semigroup S is called a <u>regular</u> <u>* semigroup</u> if it satisfies

(iii) $xx^*x = x$.

An idempotent e in S such that $e^* = e$ is called a <u>projection</u>.

The following result due to Nordahl and Scheiblich is an important property of a regular * semigroup.

RESULT 1 ([4]). <u>Let S be a regular * semigroup. Then each $\mathcal{L}$-class and each $\mathcal{R}$-class in S contain one and only one projection. Let e and f be projections in S. Then ef is an idempotent in S.</u>

Hereafter, a regular * semigroup $S(P)$ means that S is a regular * semigroup with the set of projections P.

LEMMA 2. *Let* $S(P)$ *be a regular * semigroup, and let* E *be the set of idempotents in* S. *Then* $E = P^2$. *More precisely, for any idempotent* e, *there exist projections* f *and* g *such that* $e\,\mathcal{R}\,f$, $e\,\mathcal{L}\,g$ *and* $e = fg$.

LEMMA 3. *Let* S *be a regular * semigroup. For any element* a *and any projection* e, a^*ea *is also a projection.*

A [*] congruence ν on a regular [*] semigroup S is called an *idempotent-separating* [*] *congruence* if $\nu \subset \mathcal{H}$.

THEOREM 4. *Let* μ $[\mu']$ *be the maximum idempotent-separating* [*] *congruence on a regular * semigroup* $S(P)$. *Then* $\mu = \mu' = \{(a,b) \in S\times S: a^*ea = b^*eb$ *and* $aea^* = beb^*$ *for all* $e \in P\}$.

Let $S(P)$ be a regular * semigroup. For any element a in S, let ρ_a and λ_a be mappings of P into P defined by

$$e\rho_a = a^*ea,$$
$$e\lambda_a = aea^*.$$

It is clear that $\rho_{ab} = \rho_a\rho_b$ and $\lambda_{ab} = \lambda_b\lambda_a$. Let A, B be subsets of P. A mapping $\alpha: A \to B$ is called a *partial isomorphism* if α is bijective and for $a_1, a_2, \ldots, a_n$ in A, $a_1a_2\ldots a_n \in A$ implies $(a_1\alpha)(a_2\alpha)\ldots(a_n\alpha) \in B$ and $(a_1a_2\ldots a_n)\alpha = (a_1\alpha)(a_2\alpha)\ldots(a_n\alpha)$. If there exists a partial isomorphism $\alpha: A \to B$, we say A is partial isomorphic to B, and denote it by $A \overset{p}{\cong} B$. For each e in P, let $\langle e\rangle = \{f \in P: f \leqq e\} = ePe$. Let $\mathcal{U} = \{(e,f) \in P\times P: \langle e\rangle \overset{p}{\cong} \langle f\rangle\}$ and for each $(e,f) \in \mathcal{U}$, let $T_{e,f}$ be the set of all partial isomorphisms of $\langle e\rangle$ onto $\langle f\rangle$. Let $T_P = \bigcup_{(e,f)\in\mathcal{U}} \{(\rho_e\alpha, \lambda_f\alpha^{-1}): \alpha \in T_{e,f}\}$. For convenience, we shall denote $(\rho_e\alpha, \lambda_f\alpha^{-1})$ simply by $\phi(\alpha)$. It is clear that $T_P \subset \mathcal{T}_P \times \mathcal{T}_P^*$, where $\mathcal{T}_P^*$ is the dual semigroup of $\mathcal{T}_P$.

THEOREM 5. (i) *Define a unary operation* $*: T_P \to T_P$ *by* $\phi(\alpha)^* = \phi(\alpha^{-1})$. *Then* T_P *is a regular* * *subsemigroup of* $\mathcal{T}_P \times \mathcal{T}_P^*$. *Moreover, the set of projections of* T_P *is* $\{(\rho_e, \lambda_e) : e \in P\}$, *and it is partial isomorphic to* P.

(ii) *For each* a *in* S, (ρ_a, λ_a) *is an element of* T_P. *Let* ξ *be a mapping of* S *into* T_P *defined by*

$$a\xi = (\rho_a, \lambda_a).$$

Then ξ *is a homomorphism whose kernel is the maximum idempotent-separating congruence on* S.

(iii) *For any* (e,f) *in* $\mathcal{U}$, $\alpha \in T_{e,f}$ *and* $g \in P$,

$$\phi(\alpha)^*(\rho_g, \lambda_g)\phi(\alpha) = (\rho_{(ege)\alpha}, \lambda_{(ege)\alpha}).$$

(iv) T_P *is a fundamental regular* * *semigroup*.

REFERENCES

[1] Hall, T. E., *On orthodox semigroups and uniform and anti-uniform bands*, J. Algebra, 16(1970), 204-217.

[2] Hall, T. E., *On regular semigroups*, J. Algebra, 24(1973), 1-24.

[3] Munn, W. D., *Uniform semilattices and bisimple inverse semigroups*, Quart. J. Math. Oxford (2), 17(1966), 151-159.

[4] Nordahl, T. E. and H. E. Scheiblich, *Regular* * *semigroups*, Semigroup Forum, 16(1978), 369-377.

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