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RIGHT SELF-INJECTIVE SEMIGROUPS ARE ABSOLUTELY CLOSED

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Hinkle [3] has shown that the direct product of column-monomial matrix semigroups over groups is right self-injective. The author [12] has shown that the full transformation semigroup on a set (written on the left) is right self-injective and so every semigroup is embedded in a right self-injective regular semigroup. While absolutely closed semigroup has been first studied in Isbell[7]. In Howie and Isbell [5] and Scheiblich and Moore [8] it has been shown that inverse semigroups, finite cyclic semigroups, totally division-ordered semigroups, right [left] simple semigroups and full transformation semigroups are absolutely closed. In Section 1 we shall show that every right [left] self-injective semigroup is absolutely closed. This will give another proof of that right [left] simple semigroups, finite cyclic semigroups and full transformation semigroups are absolutely closed. Using a result of [5] we shall show that the class of right [left] self-injective [regular] semigroups has the special amalgamation property. In Section 2 we shall show that a commutative separative semigroup is absolutely closed if and only if it is a semilattice of abelian groups. As its result we will obtain that every self-injective commutative separative semigroup is a semilattice of abelian groups. Using a characterization of self-injective inverse semigroups [9] we shall give a structure theorem for self-injective commutative separative semigroups. The complete proofs are omitted and will be given in detail elsewhere. Throughout this paper we freely use the terms "right S-system", "S-homomorphism", "right self-injective" and so on, which are referred to [12].

§1. Right self-injective semigroups. Let A, B be semigroups such that A is a subsemigroup of B. Then by Isbell [7] the set $\{b \in B \mid f(b) = g(b)$ for all semigroups C and for all homomorphisms $f, g : B \longrightarrow C$ such that $f|A = g|A\}$ is called the <u>dominion</u> of A in B and is denoted by $\mathrm{Dom}_B(A)$. A semigroup S is called <u>absolutely closed</u> if $\mathrm{Dom}_T(S) = S$ for all semigroups T containing S as a subsemigroup.

Result 1. ([4, Isbell's zigzag theorem]) Let T be a semigroup and S a subsemigroup of T. Then for each $d \in T$ $d \in \mathrm{Dom}_T(S)$ if and only if $d \in S$ or there exist $s_0, s_1, \ldots, s_{2m} \in S$ and $x_1, \ldots x_m, y_1, \ldots, y_m \in T$ such that $d = s_0 y_1$, $s_0 = x_1 s_1$, $s_{2i-1} y_i = s_{2i} y_{i+1}$, $x_i s_{2i} = x_{i+1} s_{2i+1}$ $(1 \leqq i \leqq m-1)$, $s_{2m-1} y_m = s_{2m}$ and $x_m s_{2m} = d$.

Theorem 1. Every right [left] self-injective semigroup is absolutely closed.

The next result follows from Theorem 1, and Corollary 1,2 of [12].

Corollary 1. I. ([8, H. Scheiblich and K. Moore]) Full transformation semigroups are absolutely closed.
II. The direct product of column [row]-momonial matrix semigroups over groups is absolutely closed.

According to [11] a semigroup S with 1 is called <u>completely right injective</u> if every right S-system is injective. It is clear that all the homomorphic images of a completely right injective semigroup are completely right injective, of course, right self-injective.

Thus we have

Corollary 2. All the homomorphic images of a completely right injective semigroup are absolutely closed.

Remark 1. It easily follows from Isbell's zigzag theorem that a semigroup S is absolutely closed if and only if S_0 [S^1] is absolutely closed, where S_0 [S^1] denotes the semigroup obtained from S by adjoining a zero [an identity]. If a semigroup S is right simple, then S_0^1 (= $(S_0)^1$) is completely right injective. Thus it follows from Corollary 2 and the above that S is absolutely closed. Also if a semigroup S is finite and cyclic then we can show that S_0^1 is a self-injective semigroup (see[12]). Hence it follows from Theorem 1 and the above that S is absolutely closed. These results have been obtained by Howie and Isbell[5].

Let $\mathfrak{A}$ be any class of algebras. According to Hall [2], if for some index set I, $\{S_i : i \in I\}$ is an indexed set of algebras from $\mathfrak{A}$ having a common subalgebra U also in $\mathfrak{A}$, then the list $(S_i : i \in I : U)$ is called an <u>amalgam from $\mathfrak{A}$</u>. If there exist an algebra W and monomorphisms $\phi_i : S_i \longrightarrow W$ $(i \in I)$ such that $\phi_i | U = \phi_j | U$ and $\phi_i(S_i) \cap \phi_j(S_j) = \phi_i(U)$ for all distinct $i, j \in I$, then the amalgam $(S_i : i \in I : U)$ is said to be <u>strongly embeddable in</u> W. If an amalgam of the form (S,S:U) from $\mathfrak{A}$ is strongly embeddable in an algebra from $\mathfrak{A}$, then U is said to be <u>closed in</u> S (within $\mathfrak{A}$). If U is closed in S within $\mathfrak{A}$ for all U, $S \in \mathfrak{A}$ with $U \subsetneq S$, then $\mathfrak{A}$ is said to <u>have the special amalgamation property</u>. If every amalgam from $\mathfrak{A}$ is strongly embeddable in an algebra from $\mathfrak{A}$, then $\mathfrak{A}$ is said to have the strong amalgamation property.

Result 2. ([4, theorem 2.4]) Let U, S be semigroups such that U is a subsemigroup of S. Then U is closed in S (within the class of semigroups) if and only if $\mathrm{Dom}_S(U) = U$.

This follows from Theorem 1, Result 2 , and Corollary 3 [12].

Theorem 2. The class of right [left] self-injective [regular] semigroups has the special amalgamation property.

The following example shows that the classof right [left] self-injective [regular] semigroups does not have the strong amalgamation property. This is constructed from an example in Imaoka [6].

Example. Let U = {0, e, f, g, 1}, V = {0, e, f, g, h, 1} and W = {0, e, f, g, x, y, 1} be semigroups whose multiplicative tables are :

U	0	e	f	g	1
0	0	0	0	0	0
e	0	e	f	g	e
f	0	e	f	g	f
g	0	e	f	g	g
1	0	e	f	g	1

V	0	e	f	g	h	1
0	0	0	0	0	0	0
e	0	e	f	g	f	e
f	0	e	f	g	f	f
g	0	e	f	g	g	g
h	0	e	f	g	h	h
1	0	e	f	g	h	1

W	0	e	f	g	x	y	1
0	0	0	0	0	0	0	0
e	0	e	f	g	x	y	e
f	0	e	f	g	x	y	f
g	0	e	f	g	x	y	g
x	0	x	y	x	x	y	x
y	0	x	y	x	x	y	y
1	0	e	f	g	x	y	1

By [11] U, V and W are completely right injective, of course, right self-injective and regular. Suppose now that the amalgam (V, W:U) is embeddable in a semigroup S. But in S we have $xh = (xe)h = x(eh) = xf = y$ and $xh = (xg)h = x(gh) = xg = x$. This is a contradiction. Hence the amalgam (V, W:U) can not be embedded in any semigroup.

§2. Commutative separative semigroups. Let S be a commutative separative semigroup. Then by [1, Theorem 4. 18] S is uniquely expressible as a semilattice Λ of archimedean cancellative semigroups S_α ($\alpha \in \Lambda$) and S can be embedded in a semigroup T which is the same semilattice Λ of abelian groups G_α ($\alpha \in \Lambda$) where G_α is the quotient group of S_α for each $\alpha \in \Lambda$, i.e. every element of G_α can be expressed in the form ab^{-1} with a and b in S_α.

Let ξ, ψ be homomorphisms of T to any semigroup W such that $\xi|S = \psi|S$. Then for each G_α, $\xi(G_\alpha)$ and $\psi(G_\alpha)$ are contained in a subgroup H of W. Hence $\xi(a^{-1}) = \psi(a^{-1})$ for all $a \in S_\alpha$. Because that both $\xi(a^{-1})$ and $\psi(a^{-1})$ are inverses of $\xi(a)$ in the group H. Then it is clear that $\xi|G_\alpha = \psi|G_\alpha$. Therefore we have $\xi = \psi$. This implies that $Dom_T(S) = T$. Thus we have

Theorem 3. Let S be a commutative separative semigroup. Then S is absolutely closed if and only if S is a semilattice of abelian groups.

In [10] we studied self-injective non-singular semigroups and showed that every self-injective non-singular semigroup is a semilattice of groups and every commutative non-singular semigroup is separative.

More generally by Teorem 1,3 we have

Theorem 4. Every self-injective separative commutative semigroup is a semilattice of abelian groups.

In [9] B. Schein characterized self-injective inverse semigroups as follows : Let S be an inverse semigroup and E_S the set of idempotents of S. A subset B of S is <u>compatible</u> if for each $b \in S$ there is $e_b \in E_S$ with $be_b = b$ and $be_c = ce_b$ for all b, $c \in B$. Define an order $\leqq$ on S by $a \leqq b$ $(a, b \in S)$ if and only if $a \in bE_S$. S is complete if every compatible set B of S has the least upper bound $\bigvee B$ relatively to $\leqq$. S is infinitely distribute if $(\bigvee B)a = \bigvee Ba$ for any compatible set B of S and for any $a \in S$. S is E_S-reflexive if $st \in E_S$ implies $st \in E_S$.

Result 3. ([9, 2.3 Theorem]) Let S be an inverse semigroup and E_S the set of idempotents of S. Then S is self-injective if and only if S is complete, infinitely distribute and E_S-reflexive.

Here we can obtain the following :

Theorem 5. Let S be a commutative semigroup. Then S is self-injective and separative if and only if S is a semilattice Λ of abelian groups G_α $(\alpha \in \Lambda)$ satisfying the followings : (1) Λ is self-injective, (2) for any set $\{g_\alpha\}_{\alpha \in X}$ such that $g_\alpha e_\beta = g_\beta e_\alpha$ $(\alpha, \beta \in X$, $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$, e_α, e_β are identities of G_α, G_β, respectively) there exists $g \in S$ such that $ge_\alpha = g_\alpha$ for all $\alpha \in X$.

REFERENCES

1. A. H. Clifford and G. B. Preston, Algebraic theory of semigroups, Vol. 1(1961), Amer. Math. Soc., Providence, R. I..
2. J. E. Hall, Amalgamation and inverse and regular semigroups, Trans. Amer. Math., 246(1978), 395-406.
3. C. Hinkle, Jr., General semigroups of quotients, Trans. Amer. Math., 183(1973), 87-117.
4. J. M. Howie, An introduction to semigroup theory, London Math.

Soc. Monographs No.7, Academic Press, London & New York, 1976.

5. J. M. Howie and J. R. Isbell, Epimorphisms and dominions, II, J. Algebra, 6(1967), 7-21.

6. T. Imaoka, Free products with amalgamation of bands, Mem. Fac. Lit. & Sci., Shimane Univ., Nat. Sci., 10(1976), 7-17.

7. J. R. Isbell, Epimorphisms and dominions, "Proc. conference on Categorical algebra, La Jolla, 1965", Springer-verlag, 1966.

8. H. E. Scheiblich and K. C. Moore, $\mathcal{J}_X$ is absolutely closed, Semigroup Forum, 6(1973), 216-226.

9. B. M. Scein, Injective monars over inverse semigroups, Colloquia Math. Soc. Janos Bolyai 20, Algebraic theory of semigroups, Szeged (Hungary), 1976, 519-544.

10. K. Shoji, Self-injective non-singular semigroups, Proc. of the 3rd Sym. on Semigroups, Atinter-university seminar house of Kansai, 1979, 69-78.

11. _______, Completely right injective semigroups, Math. Japonicae. 24(1980), 601-607.

12. ________, The injective hull of right reductive semigroups as right S-systems. In preparation.

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