数理解析研究所講究録
第 397 巻 1980 年 9-15



# How to rank the vertices in a paired comparison digraph

Mikio KANO Akio SAKAMOTO

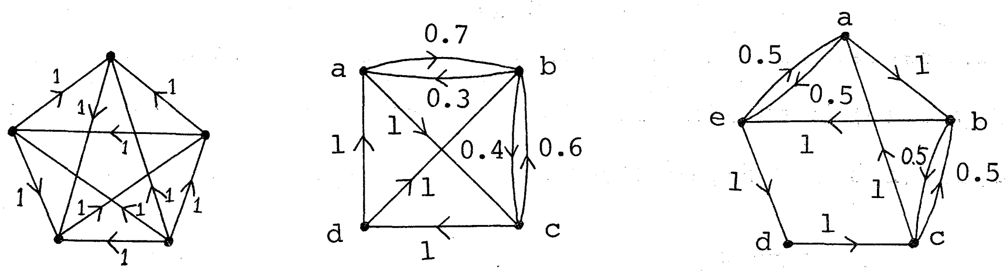
Akashi Technological College Tokushima University

Let D=(V,E) be a weighted digraph. The weight of an arc (x,y) of D is denoted by W(x,y). A weighted digraph D is called a *paired comparison digraph (p.c.digraph)* if D satisfies the following conditions;

(1) $0 < W(x,y) \leqq 1$ for every arc (x,y) of D,

(2) $W(x,y) + W(y,x) = 1$ if both (x,y) and (y,x) are arcs of D, and

(3) $W(x,y) = 1$ if (x,y) is an arc of D but (y,x) is not one of it.




Tournament (a) Complete p.c.digraph (b) Balanced p.c.digraph (c)

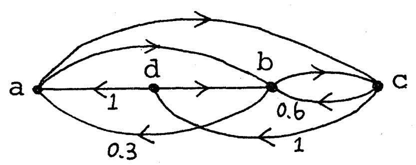
Fig.1 Paired comparison digraphs.

A tournament can be considered a paired comparison digraph if we set the weight of every arc be equal to one (see Fig.1.(a)). The p.c.digraph of Fig.1.(b) may be regarded as the results of a round-robin tournament. For instance, it is only known between player "a" and player "c" that "a" won "c", but we have more detailed information between players "a" and "b", that is, we know that "a" won "b" by 7 points to 3 or "a" had seven victories and three defeats in the games between them. The p.c.digraph of Fig.1.(c) represents that "a" won "b", the game between "a" and "d" was not played and that between "a" and "e" ended in a draw and so on.

Throughout this paper, let $D=(V,E)$ denote a paired comparison digraph and $|V|=n$. A ranking $\alpha$ is a bijection from $V$ to $\{1,2,...,n\}$. The rank of a vertex $v$ of $D$ defined by $\alpha$ is denoted by $\alpha(v)$. For a ranking $\alpha$, an arc $(x,y)$ of $D$ is called a *backward arc* of $\alpha$ if $\alpha(y)<\alpha(x)$. The set of the backward arcs of $\alpha$ is denoted by $B(\alpha)$ and its length is defined by the following.

$$\| B(\alpha) \| = \sum_{(x,y)\in B(\alpha)} W(x,y)[\alpha(x) - \alpha(y)]$$

For instance, if $\alpha$ is a ranking of a paired comparison digraph in Fig.1.(b) with $\alpha(a)=1$, $\alpha(b)=3$, $\alpha(c)=4$ and $\alpha(d)=2$, then $B(\alpha) =\{(d,a),(b,a),(c,d),(c,b)\}$ and $\| B(\alpha) \|=4.2$ (see Fig.2).




$$\| B(\alpha) \| = 1\times1 + 0.3\times2 + 1\times2 + 0.6\times1 = 4.2$$

Fig.2 Length of the backward arcs of $\alpha$.

A ranking $\alpha$ is called an *optimal ranking* of $D$ defined by the length of backward arcs if the length of the backward arcs of $\alpha$ is minimum among all the rankings. We denote the set of all optimal rankings of $D$ by $ORB(D)$ and the length of an optimal ranking of $D$ by $\|D\|^B$. That is,

$$\|D\|^B = \min_{\alpha} \|B(\alpha)\|$$

$$ORB(D) = \{\alpha \mid \|B(\alpha)\| = \|D\|^B\}$$

*Theorem 1.* *Let D be a paired comparison digraph. Then $\|D\|^B = 0$ if and only if D is acyclic.*

*Theorem 2.* *Let $D_1, D_2, \ldots, D_r$ be the strong components of a paired comparison digraph D. Then each $D_i$ is a paired comparison digraph and*

$$\|D\|^B = \|D_1\|^B + \|D_2\|^B + \ldots + \|D_r\|^B.$$

For every vertex v of D, let $K^+(v) = \sum_{(v,x)\varepsilon E} W(v,x)$ and $K^-(v) = \sum_{(x,v)\varepsilon E} W(x,v)$. A paired comparison digraph D is called to be *balanced* if $K^+(v)=K^-(v)$ for every vertex v of D (see Fig.1.(c)). A paired comparison digraph D is called *complete* if there is at least one arc between every two distinct vertices of D (see Fig.1.(b)). Of course a tournament is complete.

*Theorem 3.* *Let D be a paired comparison digraph. Then every ranking of D is optimal if and only if D is balanced and complete.*

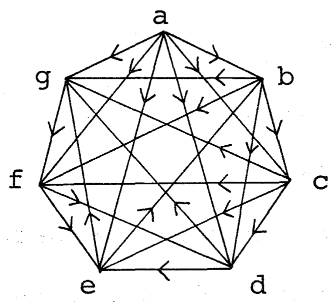
For every vertex v of D, the average of the ranks of v defined by all optimal rankings of D is called the *proper rank* of v and written by $\|v\|$. That is,

$$\|v\|_D = \frac{1}{|ORB(D)|} \sum_{\alpha \varepsilon ORB(D)} \alpha(v)$$

We shall show some properties of proper ranks.

*Theorem 4.* *Let D be a complete paired comparison digraph. Then a ranking $\alpha$ of D is optimal if and only if $K^+(\alpha^{-1}(1)) \geqq K^+(\alpha^{-1}(2)) \geqq \ldots \geqq K^+(\alpha^{-1}(n))$. The proper rank of a vertex v is given by $\|v\| = \eta + (\zeta + 1)/2$, where $\eta = \#\{x \varepsilon V \mid$*

$K^+(x) > K^+(v)\}$ *and* $\zeta = \#\{x \in V \mid K^+(x) = K^+(v)\}$.




| v | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|
| $K^+(v)$ | 6 | 4 | 4 | 2 | 2 | 2 | 1 |
| $\|\| v \|\|$ | 1 | 2.5 | 2.5 | 5 | 5 | 5 | 7 |

$|ORB(D)| = 2! \times 3! = 12$

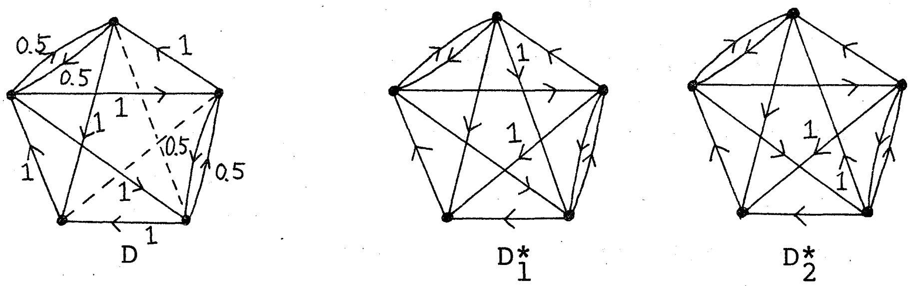
The weight of every arc is one.

Fig.3 Proper ranks of a complete p.c.digraph.

A paired comparison digraph D is called *ranking-equal* if the proper rank of every vertex is constant, that is, $\|v\| = (n+1)/2$ for every vertex v of D.

<u>*Theorem 5.*</u> *A balanced paired comparison digraph is ranking-equal.*

A paired comparison digraph D is called a *semi-complete* p.c.digraph if for every vertex v of D, there is at most one vertex which and v are nonadjacent. Let D be a semi-complete p.c.digraph. Then we make a complete p.c.digraph from D, called a *completeness* of D, by adding arcs of weight one between every nonadjacent vertices, where the initial vertex of an adding arc is one whose $K^+$ is greater than or equal to that of the other (see Fig.4). If the number of pairs of nonadjacent




Semi-complete p.c.digraph

Fig.4

Completenesses of D
$C(D) = \{D_1^*, D_2^*\}$

vertices of D whose $K^+$ are equal is $r$, then we have $2^r$ completenesses of D and denote them by C(D).

*Theorem 6.* *Let D be a semi-complete paired comparison digraph. Then*

(1) $ORB(D) = \bigcup_{D^* \in C(D)} ORB(D^*)$ *disjoint union*

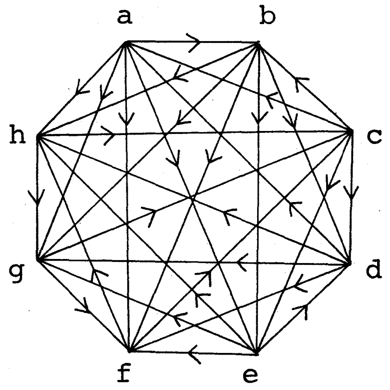
(2) $\|v\|_D = \frac{1}{|C(D)|} \sum_{D^* \in C(D)} \|v\|_{D^*}$

*Theorem 7.* *A problem to find a value* $\|D\|^B$ *of a paired comparison digraph D is NP-complete.*

Remark 1. There are some methods to rank the vertices of a tournament. One of them is a method of making use of the limit of i-th level score vector of a tournament D. The i-th level score vector $s_i$ is given by $s_i = A^i \cdot J$, where A is the adjacency matrix of D and J is a column vector of 1's. By the Perron-Frobenius theorem, the eigenvalue of A with largest absolute value is a real positive number $r$ and, furthermore,

$$\lim \frac{s_i}{r^i} = \lim \left(\frac{A}{r}\right)^i \cdot J = s$$

where $s$ is a positive eigenvector of A corresponding to r. A vertex x of D is superior to vertex y of D if the entry of s corresponding to x is greater than that corresponding to y ([1], [2],[3],[6]). This method of ranking may give good ranks in some tournaments. Though we show a remarkable example of a tournament in which the first vertex, whose rank is given by this method, has not a maximum score, where score is the number of games won by a player. This tournament is given in Fig.5.




| vertex | a | b | c | d | e | f | g | h |
|---|---|---|---|---|---|---|---|---|
| score | 5 | 5 | 4 | 4 | 4 | 2 | 2 | 2 |
| rank | 2 | 3 | 1 | 4 | 5 | * | * | * |

maximal eigenvalue 3.174...

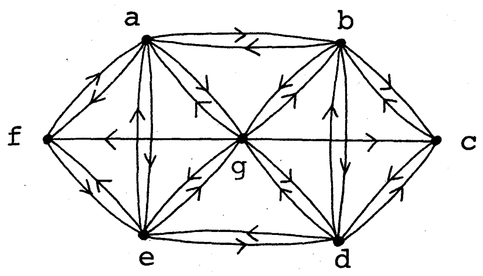
eigenvector [0.165,0.157,0.180,0.130,0.119,0.083,0.083,0.083]

Fig.5

Remark 2. At the beginning of this study, we defined the length of the backward arcs of a ranking $\alpha$ of a digraph D by the following;

$$\|B(\alpha)\| = \sum_{(x,y)\varepsilon B(\alpha)} [\alpha(x) - \alpha(y)] .$$

Hence we may consider that the weight of every arc is set to be one, even if both arcs (x,y) and (y,x) are presented in a digraph D, which represent that the game between x and y ended in a draw. We defined optimal rankings, proper ranks, ranking-equal and so on as the same in the case of paired comparison digraphs. We proved some results which are similar to theorems given in this paper and which are almost included in these theorems. In [4], we gave a counterexample of the converse proposition of theorem 5, that is, we gave a digraph which is ranking-equal but not balanced, where a digraph D is balanced if for any vertex v of D, the outdegree of v is equal to the indegree of v. This digraph is shown in Fig.6. Though if we consider this digraph as a paired comparison digraph, then this is not ranking-equal. So now we have no counterexamples of converse statement of theorem.5.




| vertex | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|
| proper rank | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| ‖v‖ | 4 | 4 | 4.75 | 4 | 4 | 4.75 | 2.5 |

Proper ranks given by old method.

x ⇄ y : Game ended in a draw.

x → y : x won y.

In paired comparison digraph: x ⇄ y (0.5, 0.5); x → y (1)

Fig.6

References

[1] Bondy,J.A.and Murty,U.S.R.; Graph theory with applications, The MacMillan Press, pp.185-189, 1976.

[2] Kendall,M.G.; Further contributions to the theory of paired comparisons, Biometrics, 11, pp.43-62, 1955.

[3] Wei,T.H.; The algebraic foundations of ranking theory, Ph.D.Thesis, Cambridge University, 1952.

[4] Sakamoto,A. and Kano,M.; A problem of how to rank the vertices in simple digraphs based on the set of backward edges, Trans. of I.E.C.E. of Japan, Vol.J62-A,No.12,pp.833-840, 1979.

[5] ———————— ; How to rank the vertices in a paired comparison digraph, in preparation.

[6] Moon,J.W.and Pulluman,N.J.; On generaliged tournament matrices, SIAM Review, Vol.12, pp.384 - 399, 1970.