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Theory and Applications of Principal Partitions of Matroids*

by

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Abstract

For these five or more years a number of applications of matroid theory have been developed in Japan to various engineering systems problems, and several novel concepts and techniques in matroid theory itself have also been introduced. The present paper will summarize those results from the author's own unifying viewpoint.

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Introduction

Matroid theory had been till some ten years ago, or may still be, regarded as one of those branches of combinatorial mathematics which are the remotest from practical applications. Indeed, it had actually been so. Since H. Whitney's initiation of the theory [49], it had been studied from purely theoretical standpoints, and the few trials that had been done for applications were, in fact, merely to rewrite in terms of matroids --- or, at best, to extend only conceptually --- some of known facts which had been, or could have been, obtained by means of graphs or linear algebra, and that in a less legible fashion [3], [33], [46].

However, there do exist several problems of systems-engineering character that can be recognized, formulated and/or solved only by the help of matroid theory, or, at least, are much easier to treat and understand by means of matroids than without matroids. Many such examples have been developed for these few years, for the most part by Japanese researchers.

From the mathematical point of view, the most useful part of matroid theory is that part which deals with the problems concerning the minimum-weight maximum-cardinality intersection of independent sets from two different matroids (or poly-matroids) and which J. Edmonds and D. R. Fulkerson began to intensively investigate about ten years ago [9], [10]. In applying that part of the theory to practical problems, we had to substantially generalize and refine the theory itself in order to get more systematic results. Thus, we have been led to a concept which is the (poly-)matroidal generalization and refinement of the "principal partition" originally introduced with respect to graphs.

For expository books and papers on the mathematics of matroids and on its connection with combinatorial optimization problems, see [3], [31], [44], [45], [48] and [50].

1. Mathematical Tools

The concepts and theorems which are of fundamental importance for our examples of applecation are as follows.

1° Submodular functions, polymatroids and matroids [9], [48], [50]: --- Let E be a finite set. A set function ρ: $2^E \to R$ (real numbers) is said to be submodular if

$$(\rho 2)\quad \rho(X \cup Y) + \rho(X \cap Y) \leqq \rho(X) + \rho(Y) \qquad \text{for any } X, Y \subseteq E. \tag{1.1}$$

If, furthermore, ρ is nonnegative and nondecreasing:

$$(\rho 0)\qquad \rho(X) \geqq 0 \qquad \text{for any } X \subseteq E, \tag{1.2}$$

$$(\rho 1)\qquad \rho(X) \geqq \rho(Y) \qquad \text{for any } X, Y \subseteq E \text{ such that } X \supseteq Y, \tag{1.3}$$

and satisfies $\rho(\emptyset) = 0$, then (E, ρ) is called a polymatroid on E with ρ as the rank function.

The nagative of a submodular function is a supermodular function, and a function which is at the same time submodular and supermodular is called a modular function. A function $x:E \to R$ may be regarded as a modular function by defining

$$x(X) = \textstyle\sum_{e \in X} x(e) \qquad \text{for every } X \in E. \tag{1.4}$$

The cardinality $|X|$ of a subset X of E determines a modular function such that $|\{x\}| = 1$ for every $x \in E$.

If the rank function ρ of a polymatroid (E, ρ) is integer-valued and satisfies

$$(\rho 0')\qquad \rho(X) \leqq |X| \qquad \text{for any } X \subseteq E, \tag{1.5}$$

then the polymatroid is called a matroid.

For a polymatroid (E, ρ), a modular function x is said to be an independent vertor if

$$x(X) \leqq \rho(X) \qquad \text{for every } X \subseteq E, \tag{1.6}$$

and an independent vector x such that x(E) is the largest possible, i.e. that $x(E) = \rho(E)$, is called a base. If a polymatroid is a matroid and if an independent vector x is integer-valued, then, by virtue of the conditions (1.5), (1.6), x is 0-1 valued and is the indicator function of a subset of E, so that integer-valued independent vectors of a matroid are identified with those subsets of E which are called independent sets of the matroid. Directly, an independent set of a matroid (E, ρ) is defined as a subset I of E such that

$$|I| = \rho(I), \tag{1.7}$$

and a base is an independent set with the greatest cardinality. The family $\mathcal{I}$ of all independent sets of a matroid satisfies the following system of axioms:

(I0) $\emptyset \in \mathcal{I}$;

(I1) if $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$;

(I2) if $I, J \in \mathcal{I}$ and $|I| < |J|$ then there is an element $x \in J - I$ such that $I \cup \{e\} \in \mathcal{I}$.

Likewise, the family $\mathcal{B}$ of all bases of a matroid satisfies the system of axioms:

(B0) $\mathcal{B} \neq \emptyset$;

(B1) no element of $\mathcal{B}$ is a proper subset of another;

(B2) for any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 - B_2$, there is a $y \in B_1 - B_2$ such that

$$(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}, \quad (B_2 - \{y\}) \cup \{x\} \in \mathcal{B}.$$

A subset of E of a matroid (E, ρ), which is not independent, is a dependent set, and a minimal (with respect to the set-inclusion) dependent set is called a circuit. The family $\mathcal{C}$ of all circuits of a matroid satisfies the system of axioms:

(C0) $\emptyset \notin \mathcal{C}$;

(C1) no element of $\mathcal{C}$ is a proper subset of another;

(C2) if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $x \in C_1 \cap C_2$, then

there is a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 - \{x\}$.

The closure cl(X) of a subset X of E of a matroid is defined as the largest superset of X such that $\rho(\mathrm{cl}(X)) = \rho(X)$. The function cl: $2^E \to 2^E$ is well-defined and satisfies the system of axioms:

(cl0) $X \subseteq \mathrm{cl}(X)$ for every $X \subseteq E$;

(cl1) if $X \subseteq \mathrm{cl}(Y)$ $(X, Y \subseteq E)$ then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;

(cl2) if $x \in \mathrm{cl}(Z \cup \{y\}) - \mathrm{cl}(Z)$ $(Z \subseteq E)$

then $y \in \mathrm{cl}(Z \cup \{x\}) - \mathrm{cl}(Z)$.

If $I \in \mathcal{I}$ and $x \in \mathrm{cl}(I) - I$, then there is a unique circuit in $I \cup \{x\}$ which is expressed as

$$C \equiv C(x|I) = \{ y \mid (I \cup \{x\}) - \{y\} \in \mathcal{I}\}. \tag{1.8}$$

For a matroid $M = (E, \rho)$, there is another matroid $M^* = (E, \rho^*)$ on the same set such that every base of the former is the complement of a base of the latter, and vice versa. The latter M* is called the dual of the former M. Obviously, M is the dual of M*. The rank functions of the dual pair of matroids are connected with each other by the relation:

$$|X| - \rho^*(X) = \rho(E) - \rho(E - X). \tag{1.9}$$

The reduction of a matroid $M = (E, \rho)$ to a subset U of E is the matroid

$$M|U = (U, \rho|_U), \tag{1.10}$$

where $\rho|_U$ is the restriction of ρ to 2^U. Sometimes, we senote M|U by $M - (E - U)$, and say that M|U is obtained from M by deleting (or reducing) subset E - U. The family $\mathcal{I}|_U$ of independent sets of $M|U$ is simply the subfamily $\mathcal{I}|_U = \{ I \mid I \in \mathcal{I}, I \subseteq U\}$ of the family $\mathcal{I}$ of independent sets of M. The <u>contraction</u> of a matroid $M = (E, \rho)$ to a subset U of E is the matroid

$$M \times U = (M^*|U)^* = (U, \rho_U). \tag{1.11}$$

The rank function ρ_U of $M \times U$ is expressed as

$$\rho_U(X) = \rho(X \cup (E - U)) - \rho(E - U) \quad \text{for } X \subseteq U. \tag{1.12}$$

Sometimes, we denote $M \times U$ by $M \div (E - U)$, and say that $M \times U$ is obtained from M by <u>contracting</u> subset $E - U$. A matroid obtained from M by a sequence of operations of reduction and contraction is called a <u>minor</u> of M.

For a family $\mathcal{F}$ ($\subseteq 2^E$) of subsets of E and a map ϕ: $E \to E'$, the family $\phi(\mathcal{F})$ of subsets of E' is defined by

$$\phi(\mathcal{F}) = \{ \phi(X) \mid X \in \mathcal{F}\}. \tag{1.13}$$

Then, for a matroid $M = (E, \rho)$ with the family $\mathcal{I}$ of independent sets and a map ϕ: $E \to E'$, there is a matroid $M = (E', \rho')$, called the <u>image</u> of M under ϕ and uniquely determined from M and ϕ, such that M' has $\phi(\mathcal{I})$ as the family of independent sets. The rank function ρ' of M' is expressed as

$$\rho'(X) = \min \{ \rho(\phi^{-1}(Y)) + |X - Y| \mid Y \subseteq X\} \quad \text{for } X \subseteq E. \tag{1.14}$$

For two matroids $M_1 = (E_1, \rho_1)$ and $M_2 = (E_1, \rho_2)$ with the families $\mathcal{I}_1$, $\mathcal{I}_2$ of independent sets, respectively, their <u>union</u> $M = (E_1 \cup E_2, \rho)$ is defined as the matroid whose family $\mathcal{I}$ of independent sets is expressed as

$$\mathcal{I} = \{ I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2 \}. \tag{1.15}$$

The rank function ρ of the union is determined from ρ_1 and ρ_2 by the equation:

$$\rho(X) = \min \{\rho_1(Y) + |X - Y| \mid Y \subseteq X\}$$
$$= \max \{\rho_1(Y) + \rho_2(X - Y) \mid Y \subseteq X\} \quad \text{for } X \subseteq E . \tag{1.16}$$

The concepts of dual, reduction, contraction, image and union are naturally generalizes to polymatroids.

2° A variant of the Jordan-Hölder theorem for lattices?---- Although the Jordan-Hölder theorem is not stated in ordinary textbooks on algebra in the form in which we shall make use of it in the following, the fact itself will be evident for those who are skilled in the art [22].

Let E be a finite set and L be a family of subsets of E which is closed under the operations of taking union and intersection: if X, Y $\in L$ then $X \cup Y \in L$ and $X \cap Y \in L$. (L is thus a distributive lattice.) Then, L has both the minimum $E^- = \bigcap_{X\in L} X$ and the maximum $E - E^+ = \bigcup_{X\in L} X$. Every maximal chain $E^- \equiv X_0 \subsetneqq X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_{h-1} \subsetneqq X_h \equiv E - E^+$ ($X_i \in L$) has not only the same length, but the partition of E into blocks:

$$E = E^- \cup [\bigcup_{i=1}^{h} (X_i - X_{i-1})] \cup E^+ \tag{1.17}$$

is independent of the choice of the chain. The family of blocks in the middle term of the right-hand side of (1.17) will be denoted by F:

$$E = E^- \cup [\bigcup_{W\in F} W] \cup E^+. \tag{1.18}$$

A partial order $\succeq$ can be introduced in F by defining $W_1 \succeq W_2$ ($W_1, W_2 \in F$) if and only if $W_2 \subseteq X$ ($\in L$) whenever $W_1 \subseteq X$ ($\in L$) . With respect to this partial order, a <u>monotone</u> <u>dissection</u> of F is defined as a partition of F into two blocks: $F = F^+ \cup F^-$ ($F^+ \cap F^- = \emptyset$) such that there are no two elements $W_1 \in F^+$ and $W_2 \in F^-$ which satisfy $W_2 \succneqq W_1$. Then, for every monotone dissection of F into

F^+ and F^-, $X = E^- \cup \bigcup_{W \in F^-} W$ is an element of F, and, conversely, every element of L is expressed in this form, uniquely.

In the above sense, either L or F represents essentially the same structure of a family of subsets of E, but, in general, the latter representation is far simpler and more suitable for practical manipulation than the former.

3° Principal partition: --- The origin of the concept of principal partition, which we are now intensively making use of in the structural analysis of various practical engineering systems, may be traced back to the decomposition theory [5], [6], [7] for bipartite graphs by Canadian mathematicians A. L. Dulmage and N. S. Mendelsohn, and the term "principal partition" was first used by G. Kishi and Y. Kajitani [28] for a decomposition of a single graph into three parts. These ideas have been extended and refined in a number of directions, and have been given a formulation from the unifying viewpoint by M. Iri [21]. The formalism adopted in the present paper follows that in [21] with detailed results worked out by M. Nakamura in his master's thesis [34a].

If we are given a submodular function $\rho: 2^E \to R$ together with a modular function $w: 2^E \to R$ which has positive value except for $w(\emptyset) = 0$ (in most cases, w is the cardinality function), then we may consider the problem of finding the subsets X of E which minimize

$$P_\lambda(X) \equiv \rho(X) - \lambda w(X) \qquad (X \subseteq E) , \tag{1.19}$$

where λ is a real parameter running from $-\infty$ to $+\infty$. If X_1 is a solution to the problem with parameter λ_1 and X_2 is a solution to the problem with parameter λ_2, where $\lambda_1 \geqq \lambda_2$, then simple calculation will yield the inequality:

$$P_{\lambda_1}(X_1 \cup X_2) + P_{\lambda_2}(X_1 \cap X_2)$$
$$\leqq P_{\lambda_1}(X_1) + P_{\lambda_2}(X_2) - (\lambda_1 - \lambda_2)\, w(X_2 - X_1) \; . \qquad (1.20)$$

Since $P_{\lambda_1}(X_1) \leqq P_{\lambda_1}(X_1 \cup X_2)$, $P_{\lambda_2}(X_2) \leqq P_{\lambda_2}(X_1 \cap X_2)$ and $0 \leqq (\lambda_1 - \lambda_2) w(X_2 - X_1)$, we should have

$$P_{\lambda_1}(X_1) = P_{\lambda_1}(X_1 \cup X_2), \quad P_{\lambda_2}(X_2) = P_{\lambda_2}(X_1 \cap X_2), \quad \text{and} \quad (\lambda_1 - \lambda_2) w(X_2 - X_1) = 0, \qquad (1.21)$$

i.e. $X_1 \cup X_2$ should be a solution to the problem with parameter λ_1, and $X_1 \cap X_2$ a solution to the problem with parameter λ_2, and, if $\lambda_1 > \lambda_2$, then $w(X_2 - X_1) = 0$ or $X_2 \subseteq X_1$. Therefore, if we denote the family of all the subsets of E which are solutions to the problem with parameter λ by $L(\lambda)$, the $L(\lambda)$ is closed under the union and intersection operations, and, furthermore,

$$L = \bigcup_\lambda L(\lambda) \qquad (1.22)$$

is also closed under those operations. It is not difficult to see from these properties of the minimizing solutions of (1.19) that the partially ordered sets $F(\lambda)$ associated with $L(\lambda)$ and F associated with L are related to one another as

$$F = \bigcup_\lambda F(\lambda), \qquad (1.23)$$

that if $\lambda_1 > \lambda_2$ then, for any $W_1 \in F(\lambda_1)$ and any $W_2 \in F(\lambda_2)$, we have $W_1 \not\geq W_2$, and that

$$E = \bigcup_\lambda \bigcup_{W \in F(\lambda)} W. \qquad (1.24)$$

What is most important is the fact that, since everything is finite, there are only a finite number of values of λ for which $F(\lambda)$ is nonempty. We shall call those values the <u>critical</u> values. Thus, the unions in (1.22), (1.23) and (1.24) are essentially taken over a finite set Λ of critical values of λ.

The partition (1.24) of E with a partial order among its blocks is the <u>principal partition</u> with respect to ρ (and w).

If we take the edge set of a graph for E, the rank of the subgraph (=partial graph) generated by a subset X of edges for $\rho(X)$, and $|X|/2$ for $w(X)$, then we shall have N. Tomizawa's refinement [41] (what is essentially the same is found also in H. Narayanan's [34b]) of Kishi and Kajitani's partition of a graph [28].

The most general and unified treatment of the principal parttion will be for the structure which consists of a bipartite graph $G = (U,A,V)$ (U and V being the left vertex set and the right, and A being the edge set) and two polymatroids $M_1 = (U, \rho_1)$ and $M_2 = (V, \rho_2)$ on its two vertex sets. We shall denote this structure by $(\rho_1|G|\rho_2) = (\rho_1|U, A, V|\rho_2)$ in the following.

We consider the covers (X, Y) of G, i.e. those pairs of subsets $X \subseteq U$ and $Y \subseteq V$ of vertices which cover all the edges: $A \subseteq X \times Y$. The family of all the covers is, as is well known, closed under the operations of union: $(X_1, Y_1) \cup (X_2, Y_2) = (X_1 \cup X_2, Y_1 \cap Y_2)$ and of intersection: $(X_1, Y_1) \cap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cup Y_2)$. Among covers, those for which

$$P_\lambda(X, Y) = (1 + \lambda)\, \rho_1(X) + (1 - \lambda)\, \rho_2(Y) \qquad (1.25)$$

takes the minimun value play an important role, where λ is a parameter running from -1 to 1. We shall call those covers the <u>minimum-rank covers</u> (or, simply, <u>minimum covers</u>). If (X_1, Y_1) is a minimum cover with parameter λ_1, and (X_2, Y_2) a minimum cover with parameter λ_2, then it is a staightforward calculation to derive the inequality:

$$P_{\lambda_1}(X_1 \cap X_2, Y_1 \cup Y_2) + P_{\lambda_2}(X_1 \cup X_2, Y_1 \cap Y_2)$$

$$
\begin{aligned}
&\leqq P_{\lambda_1}(X_1, Y_1) + P_{\lambda_2}(X_2, Y_2) \\
&\quad -(1+(\lambda_1+\lambda_2)/2)(\rho_1(X_1)+\rho_1(X_2)-\rho_1(X_1\cup X_2)-\rho_1(X_1\cap X_2)) \\
&\quad -(\lambda_1-\lambda_2)/2\cdot(\rho_1(X_1)-\rho_1(X_1\cap X_2)+\rho_1(X_1\cup X_2)-\rho_1(X_2)) \\
&\quad -(1+(\lambda_1+\lambda_2)/2)(\rho_2(Y_1)+\rho_2(Y_2)-\rho_2(Y_1\cup Y_2)-\rho_2(Y_1\cap Y_2)) \\
&\quad -(\lambda_1-\lambda_2)/2\cdot(\rho_2(Y_2)-\rho_2(Y_1\cap Y_2)+\rho_2(Y_1\cup Y_2)-\rho_2(Y_1)) \\
&\leqq P_{\lambda_1}(X_1, Y_1) + P_{\lambda_2}(X_2, Y_2) \\
&\quad -(\lambda_1-\lambda_2)/2\cdot\{(\rho_1(X_1)-\rho_1(X_1\cap X_2))+(\rho_1(X_1\cup X_2)-\rho_1(X_2)) \\
&\qquad +(\rho_2(Y_2)-\rho_2(Y_1\cap Y_2))+(\rho_2(Y_1\cup Y_2)-\rho_2(Y_1))\}.
\end{aligned}
\tag{1.26}
$$

By an argument similar to the foregoing, we can conclude that, if $\lambda_1 \geqq \lambda_2$, $(X_1\cap X_2, Y_1\cup Y_2)$ is a minimum cover with parameter λ_1 and $(X_1\cup X_2, Y_1\cap Y_2)$ is a minimum cover with parameter λ_2, and that, if $\lambda_1>\lambda_2$,

$$
\begin{aligned}
&X_1 \subseteqq cl_1(X_1\cap X_2), \quad X_1\cup X_2 \subseteqq cl_1(X_2), \\
&Y_2 \subseteqq cl_2(Y_1\cap Y_2), \quad Y_1\cup Y_2 \subseteqq cl_2(Y_1).
\end{aligned}
\tag{1.27}
$$

Thus, we have a distributive lattice $L(\lambda)$ of the family of all the minimum-rank covers, for each value of λ, and hence a partially ordered set $F(\lambda)$ (each of whose element is a pair (S, T) of a subset S of U and a subset T of V). The union of $L(\lambda)$'s for all λ's also forms a distributive lattice, and the associated partially ordered set F may be expressed as $\bigcup_\lambda F(\lambda)$. This time, however, $F(\lambda)$'s for different λ's are not in general disjoint, but, as before, there are only a finite number of critical values of λ.

As special cases of the problem of this bipartite structure, we have various problems already studied to some extent. For example, the Dulmage-Mendelsohn decomposition of a bipartite graph [6] corresponds to the case where both M_1 and M_2 are free matroids (i.e. $\rho_1(X) = |X|$ and $\rho_2(Y) = |Y|$) and $\lambda = 0$. Tomizawa's partition of a matroid [41] corresponds to the case where $U = V = E$, $A = \Delta_E$ (the diagonal set of $E \times E$), $M_1 = (U, \rho_1) = (E, \rho)$ and $M_2 = M_1{}^*$.

For the algorithmic approach to the principal partition, the dual problem is important, as is the case for many other problems of mathematical-programming character. The problem dual to that of finding the minimum covers of $(\rho_1|U, A, V|\rho_2)$ is to find a flow $\xi\colon A \to R$ which

$$\text{maximizes } \sum_{a\in A} \xi(a) \tag{1.28}$$

subject to the conditions:

$$\xi(a) \geqq 0 \qquad \text{for every } a \in A, \tag{1.29}$$

$$\sum_{a\in\delta X} \xi(a) \leqq (1+\lambda)\,\rho_1(X) \qquad \text{for every } X \subseteq U, \tag{1.30}$$

$$\sum_{a\in\delta Y} \xi(a) \leqq (1-\lambda)\,\rho_2(Y) \qquad \text{for every } Y \subseteq V, \tag{1.31}$$

where δX (or δY) means a set of edges incident to the vertices of X (or Y). We may consider a more general problem of finding a ξ which, if the solution of the above problem is not unique, minimizes

$$\sum_{a\in A} w(a)\xi(a) \tag{1.32}$$

with respect to a given weight function $w\colon A \to R$.

If M_1 and M_2 are free matroids and $\lambda = 0$, then we have the maximum matching problem (or the assignment problem in the weighted case) on a bipartite graph. If M_1 and M_2 are matroids and $\lambda = 0$, we have the independent matching (or assignment) problem [26]. The well-known matroid intersection problem (see, e.g., [30] and [48]) of finding, for given two matroids $M_1 = (E, \rho_1)$ and $M_2 = (E, \rho_2)$

on the same set E, a subset of maximum cardinality which is independent both in M_1 and in M_2 is simply the case where $U = V = E$, $A = \Delta_E$, and $\lambda = 0$.

If ρ_1 and ρ_2 are nonnegative modular functions, the problem is no other than the Hitchcock-type transportation problem.

A number of algorithms are available for solving the problem of this kind. If ρ_1 as well as ρ_2 is an integral multiple of the rank function of a matroid, we can make use of the algorithm proposed in [26], and, for the genaral case, S. Fujishige's algorithm [13] works. It should be noted that the concept of "auxiliary graph" is very powerful to develop computationally efficient algorithm for this kind of poblems [20], [26], [42]. The determination of the critical values of the parameter λ requires a trick, due originally to Tomizawa [41], which is suggested in [21].

In general, the solution of the entire dual problem is decomposed into separate solutions of the subproblems $(\tilde{\rho}_{1i}|U_i, A_i, V_i|\tilde{\rho}_{2i})$, where (U_i, V_i) is an element of $\mathcal{F}$, $A_i = A \cap (U_i \times V_i)$, $M_{1i} = (U_i, \tilde{\rho}_{1i})$ is the polymatroid obtained from $M_1 = (U, \rho_1)$ by deleting all the U-blocks which are lower in the partial order than U_i and contracting all the other U-blocks except for U_i itself, and $M_{2i} = (V_i, \tilde{\rho}_{2i})$ is the polymatroid obtained from $M_2 = (V, \rho_2)$ by deleting all the V-blocks which are upper in the partial order than V_i and contracting all the other B-blocks except for V_i itself. Furthermore, if (U_i, V_i) belongs to $\mathcal{F}(\lambda)$, then

$$\tilde{\rho}_{1i}(U_i) : \tilde{\rho}_{2i}(V_i) = (1 - \lambda) : (1 + \lambda). \tag{1.13}$$

2. Examples of Problems for Which Matroids and Polymatroids are Useful

The following problems are examples to which the concepts of (poly-)matroids, especially those connected with the principal partition, are applied effectively. They will show how powerful the technique of principal partition is for a wide variety of systems problems appearing in engineering science and how clearly the relationship among seemingly different problems is revealed in the light of matoids. Moreover, it is expected that a number of novel applications of practical use and importance will be found in the near future.

2.1. Spanning arborescences

The problem of finding a minimum-weight spanning tree on a graph (whose edges are given real weights) is well known and quite a few efficient solution algorithms have been proposed [11], [29], [37], [52]. The problem is a special case of that of finding a minimum-weight base of a matroid whose elements are given weights [47], and the simplest-minded algorithm, which J. Edmonds called the "greedy algorithm", affords the solution.

In contrast with this, the directed version of the problem, i.e. the problem of finding a minimum-weight arborescence, with the root prescribed or not, on a given (directed) graph is much harder. (An arborescence with root v is a spanning tree such that, for every vertex u other than v, the path from v to u which consists of edges of the tree contains all the edges on it in the positive direction.) In western countries, Edmonds' paper [8] and its improvement by R. M. Karp [27] are usually referred to in this context, but it can hardly be said that the problem was treated simply and elegantly there. (It should be noted that, in Japan, R. Manabe and S. Kotani [32] published a fairly simple method for solving it in 1973, and that the paper [3a] published in 1965 by Chinese mathematicians treated the problem most elegantly and gave a simplest method of solution.)

If the problem is viewed in the light of matroids, it will readily be seen that any algorithm for the minimum-weight independent assignment problem or the weighted matroid intersection problem can slove it. In fact, a subset X of edges of graph G = (U, E) (with vertex set U and edge set E) is an arborescence if and only if it is a base of the circuit matroid of the graph $M_1 = (E, \rho_1)$ (where ρ_1 is the function which assigns a subset of edges the rank of the subgraph they form), and at the same time, it is independent in the matroid $M_2 = (E, \rho_2)$ where ρ_2 is the function assigning a subset of edges the number of vertices at which they end. (If a vertex u is prescribed as the root, it is not counted by ρ_2.) (It is not difficult to see that ρ_1 as well as ρ_2 satisfies the axioms (ρ0) ~ (ρ2) in 1° of §1.) Therefore, if a minimum-weight maximum-cardinality common independent set of M_1 and M_2 forms a spanning tree on G, then it is a required arborescence, and, otherwise, there is no spannig arborescence on G.

The matroidal algorithms applied to this intersection problem are as simple as those algorithms devised specially for this problem without resorting to matroids.

What will result if we apply the principal-partition technique to this problem? It has recently been investigated to some extent by Nakamura and Iri [34c]. To mention some of their results, a spanning arborescence exists if and only if all the critical values of the parameter λ is nonnegative; those parts of the graph which correspond to the blocks of $F(\lambda)$'s with negative λ are responsible for the nonexistence of the spanning arborescence where the magnitude of λ indicates how bad the condition is. When G has a spanning arborescence, the principal partition is a refinement of the decomposition of G into strongly connected components. The sigleton blocks of $F(0)$ with rank 1 are those edges which are contained in every spanning arborescence, whereas the singleton blocks with rank 0 are never used for forming a spanning arborescence.

2.2. Principal partition of a bipartite graph without matroids

This is obviously the extension of the Dulmage-Mendelsohn decomposition of a bipartite graph (U, A, V), and has the obvious application to the structural analysis of a large sparse system of linear and nonlinear equations, where U is the set of equations, V the set of variables, and A denotes the occurrences of variables in equations. The principal partition classifies the equations and the variables according to the degree of indeterminacy; i.e., the underdeterminacy takes place in the blocks of $F(\lambda)$'s with positive λ, the magnitude of λ indicating the degree of underdeterminacy, whereas the overdeterminacy takes place in the blocks of $F(\lambda)$'s with negative λ, the magnitude of λ indicating the degree of overdeterminacy. Furthermore, if there is only one critical value of λ which is equal to zero, then the system is "structurally" well posed, and the partial order among the blocks of $F(0)$ affords us the information according to which we may reduce the solution of the entire system to a series of solutions of subsystems.

Besides the obvious application such as the above, we have a curious application. We consider the field data on the usage of a language by some group of people: they consist, for example, of the set of symbols (words), the set of objects (or collections of objects), and the denotation of objects by symblols, i.e. the information about which symbols denote which (collections of) objects. We interpret the set of symblols as the vertex set U, the set of (collections of) objects as the vertex set V, and the relation of denotation as the edge set A, and we principally partition the bipartite graph (U, A, V). Then, the set U as well as V is automatically partitioned into blocks among which a partial otder is determined. This mathematical decomposition is shown to have many desirable properties which the linguists require the method of extracting the "concepts" from the observational data of language users, or of categorizing words in a hierarchical structure, to have. Therefore, this technique, if combined with some auxiliary means of preprocessing the data and postprocessing the results, will be of use for automatic extraction of concepts and for automatic construction of a thesaurus [40].

2.3. Minimum fundamental equations for a linear system

The analysis of a (linear or nonlinear) physical system such as an electric network and an elastic structure usually begins with setting up a fundamental system of equations for the system. There are two kinds of physical variables associated with a system, one called "intensive variables" and the other "extensive variables". In an electric network, intensive variables are "currents", and extensive variables are "voltages". In an elastic structure, the former are "forces", "moments (bending and/or torsional)", etc., and the latter are "displacements", "elongations", "deflections", "bendings", "torsions", etc. These variables are subject to two kinds of constraints, which together determine --- usually uniquely --- the values of the variables. The constraints of the first kind are topological or geometrical, and are linear equations to be satisfied by intensive variables alone (Kirchhoff's current law, equilibrium conditions for forces and moments) and those to be satisfied by extensive variables alone (Kirchhoff's voltage law, compatibility conditions for deformations). These equations are determined by the purely topological/geometrical structure of the system, and the set of constraints among intensive variables and that among extensive variables are "contragredient" to each other. The constraints of the other kind represent the physical properties of the elements constituting the system, and are ordinarily called the "constitutive equations". They have nothing to do with the topological /geometrical structure of the system, but depend only on the physical properties of the elements. The system is linear or nonlinear according as its constitutive equations are linear or nonlinear.

The fundamental equations of the intensive-variable type (such as the loop or mesh equations for an electric network) are set up as follows. To begin with, a minimal set of intensive variables is chosen such that all the intensive variables may be expressed as their linear combinations by means of the topological/geometrical constraints among the intensive variables. Then, the constitutive equations are used to express all the extensive variables as functions of the chosen intensive bariables. Finally, substitution in the topological/geometrical constraints among the extensive variables yields a system of equations with the chosen set of intensive variables as the unknowns. It is noteworthy that the size of the system of equations of this type, which is, of course, equal to the number of chosen intensive variables, is determined by the topological/geometrical structure of the system alone, and does not depend on the choice of unknowns. In the case of an electric network, the size is equal to the "nullity" of the graph representing the topological structure of the circuit diagram of the network, and in the case of an elastic structure, it is sometimes called the degree of "statical indeterminacy" of the structure. Similar statements obtain for the fundamental equations of the extensive-variable type (such as the nodal equations or the cut-set equations for an electric network). The size of the system of equations of this latter type does not depend on the choice of the variables either, and it is equal, e.g. in the case of an electric network, to the "rank" of the underlying graph.

There are fundamental equations of the third type, i.e. the so-called "hybrid" or "mixed" equations, where some of the intensive variables and some of the extensive variables are chosen as the unknowns of the equations. Unknowns are chosen in such a way that, for each element (or each degree of freedom of elements) of the system, either the intensive or the extensive variable associated with it may be expressed as a linear combination of the chosen variables. Unlike fundamental equations of the "pure" type, the size of the hybrid system of equations, or the number of unknowns, does depend on the choice of unknowns. Thus, there arises the problem of how to find the minimum-size system of fundamental equations of hybrid type.

To be specific, let us consider an electric network. One current variable and one voltage variable are associated with each edge of the underlying graph $G = (U, E)$. The circuit matroid $M = (E, \rho)$ is defined on the edge set E of G as in §2.1. For an arbitrary dissection (E_1, E_2) of E ($E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$), if the extensive variables, voltages, across the edges of a base of $M|E_1$ are chosen as unknowns, then the voltages across all the edges of E_1 are expressed as linear combinations of the chosen voltages. If the intensive variables, currents, in the edges of a base of $(M \times E_2)^*$ are chosen as unknowns, then the currents in all the edges of E_2 are expressed as linear combinations of the chosen currents. Thus, we have a hybrid system of equations with

$$\rho|_{E_1}(E_1) + \rho_{E_2}^*(E_2) \tag{2.1}$$

unknowns and as many equations.

Here arises the question: What dissection (E_1, E_2) makes (2.1) smallest, and how can we get it? Obviously, the question is answered by finding a subset E_1 of E which is a solution of the following equation:

$$2\rho(E_1) - |E_1| = \min \{ 2\rho(X) - |X| \mid X \subseteq E \}, \tag{2.2}$$

since it can be shown without difficulty from the definitions of reduction, contraction and dual that

$$\begin{aligned} \rho|_{E_1}(E_1) &= \rho(E_1), \\ \rho_{E_2}^*(E_2) &= \rho(E_1) - |E_1| + (|E| - \rho(E)), \end{aligned} \tag{2.3}$$

and since $|E| - \rho(E)$ is constant. As was already noted in 3° of §1, the problem of finding a subset E_1 minimizing (2.2) is equivalent to finding a minimum cover of $(\rho|E, \Delta_E, E|\rho^*)$.

Historically, the minimum-fundamental-equation problem was solved for the electrical networks by means of graphical techniques by G. Kishi and Y. Kajitani [28] and T. Ohtsuki et al. [35], and for the more general systems by means of linear algebra combined with some techniques from combinatorics by M. Iri [17] --- almost simultaneously. However, at that time, they did not note the relation to matroids, and their methods were considerably complicated. As has been demonstrated, a little knowledge about matroids will enable us to understand the essence of the problem and the principal partition will afford practically efficient algorithms for choosing unknowns. In fact, the elements of $L(0)$, and only those, associated with $(\rho|E, \Delta_E, E|\rho^*)$ are the solutions of (2.2).

2.4. Topological conditions for the existence of the unique solution in an electric network

In the analysis of an electric network with mutual couplings among its branches (=edges), the problem of the following kind arises. For the sake of simplicity, we shall consider a linear electric network whose circuit diagram is represented by the graph $G = (U, E)$ and whose branch characteristics are given in terms of self- and mutual admittances. The fundamental system of equations of the voltage-variable type, then, has the coefficient matrix of the from:

$$DYD^T \qquad (2.4)$$

where D is the fundamental cutset matrix of G (with columns corresponding to edges) and the (κ, λ)-entry $y^{\kappa\lambda}$ of Y is the admittance from branch (=edge) λ to branch κ. Here it is noted that the matrix D (as well as its transpose D^T) is determined from the topological (i.e. graphical) structure of the network without ambiguity, whereas the numerical values of $y^{\kappa\lambda}$'s are observed or measured physical quantities contaminated with various kinds of noises. Therefore, it will be admitted to assume that the "nonvanishing" values of $y^{\kappa\lambda}$'s satisfy no algebraic equation with integer coefficients, i.e. that the "nonvanishing" $y^{\kappa\lambda}$'s are "general (or generic)" over the ring or field to which the entries of the matrix D belong. The network has the unique solution if and only if $\det(DYD^T) \neq 0$. Under the above assumption, this condition is equivalent to the following condition:

There are two sets I and J ($\subseteq$ E) of edges such that

(i) the subdeterminant of D with all the rows and those columns which correspond to I does not vanish,

(ii) the subdeterminant of D with all the rows (i.e. all the columns of D^T) and those columns which correspond to J does not vanish,

and

(iii) the subdeterminant of Y with the rows corresponding to I and the columns corresponding to J does not vanish.

Since the matroid $M = (E, \rho)$ (where $\rho(X)$ ($X \subseteq E$) represents the rank of the submatrix of D consisting of the columns corresponding to X and of all the rows) coincides with the circuit matroid of G, the conditions (i) and (ii) are equivalent further to

(i') I is a spanning tree on G,

(ii') J is a spanning tree on G,

and (iii) to

(iii') there is a one-to-one correspondence between I and J such that $y^{\kappa\lambda} \neq 0$ if $\kappa(\in I)$ is in correspondence with $\lambda(\in J)$.

It is ready to formulate the problem now under consideration in the form of finding a maximum-cardinality independent matching on the bipartite structure

$(\rho|E, A, E|\rho)$ with the two "vertex" sets both being identical to the edge set E of the graph and the "edge" set A representing the existence of nonvanishing self- or mutual admittance between the edges (corresponding to a left and a right vertex) of E, and the same circuit matroid of G attached to the left and the right vertex set E. If the maximum-cardinality independent matching has as many elements as the rank of G, then $\det(DYD^T) \neq 0$, and otherwise $\det(DYD^T) = 0$.

The above formalism is essentially due to Tomizawa and Iri [25], [42]. T. Ozawa could derive a purely graphical method for solving the problem [36] with some observations on the structure which is self-evident from the standpoint of principal partition. His method made use of the artificially introduced concepts of voltage graph and current graph and did not give clear insight in the connection with other related problems. A. Recski of Hungary [38], [38a] (see also the references cited there) and B. Petersen of Denmark [36a] have investigated similar problems from a similar matroidal standpoint and have obtained similar results under somewhat more general circumstances.

If the principal partition is applied to this problem, it can be seen wherefrom the inconsistency comes and how large the over/underdeterminacy is at such and such places when $\det DYD^T = 0$, and when $\det DYD^T \neq 0$, the decomposition of the entire system into subsystems is given in terms of the blocks of $F(0)$ and the partial order among them [34b]. Thus, viewed from our standpoint, the analysis of this kind of problem proceeds in quite the same way as that which we explained in §2.2.

2.5. Order of complexity of a linear electric network

We shall use the same terminology and notation as in §2.4. In the theory of linear time-invariant lumped-constant electric networks, it is well known that the number of independent eigenmodes (with nonzero frequency) of the dynamical performance of a network (which is called the order of complexity of the network) is equal to the difference between the maximum p_{max} and the minimum p_{min} of the exponents to the time-differentiation operator in the expansion of

$$\det (DYD^T) = \sum_{p=p_{min}}^{p_{max}} c_p \left(\frac{d}{dt}\right)^p , \qquad (2.5)$$

where the admittances $y^{\kappa\lambda}$ are regarded as integro-differential operators with respect to the time t which are one of the forms $c\frac{d}{dt} = c\left(\frac{d}{dt}\right)^1$, $c = c\left(\frac{d}{dt}\right)^0$ and $c\int dt = c\left(\frac{d}{dt}\right)^{-1}$. If $\det(DYD^T) \neq 0$ (see the previous subsection), there is at least one nonvanishing c_p. A nonvanishing term in the expansion corresponds to a maximum-cardinality independent matching on the bipartite structure $(\rho|E, A, E|\rho)$ in §2.4, and the exponent to the differentiation operator of the term is equal to the sum of the exponents to the differentiation operators of the admittances represented by the edges of that matching.

Thus, the order-of-complexity problem is evidently reduced to the independent assignment problem on the bipartite graph where the weight of an edge is put equal to (the negative of) the exponent to the differentiation operator of the admittance the edge represents. The solution algorithm is ready.

In the light of matroids, the formulation and solution of the order-of-complexity problem was immediate [25], [43], but, "without matroids" it had long remained to be a difficult unsolved problem for electric network theorists.

Needless to say that, in applying the principal partition, only $F(0)$ will appear, and that the entire problem is decomposed into the subproblems corresponing to the blocks of $F(0)$.

Recently, Petersen proposed a systematic method for finding not only the order of complexity but also for seeing which of the coefficients c_p's (in the denominator $\det(DYD^T)$ as well as in the numerator which is a certain subdeterminant of DYD^T) of a transfer function do not vanish [36b].

2.6. Existence of a hybrid immittance matrix of a general linear n-port

Y. Oono once posed the following problem [35a], which is one of the most fundamental problems in electric network theory. In general, a linear electrical n-port is defined as a device with n "ports", with each of which one voltage variable u_κ and one current variable i_κ are associated ($\kappa = 1, \cdots, n$). The characteristic of an n-port is described by an $n \times 2n$ matrix $[P|Q]$ of rank n, which is the coefficient matrix of the system of homogeneous linear equations to be satisfied by port-voltages u_κ's and port-currents i_κ's :

$$\sum_{\kappa=1}^{n} P_\alpha^\kappa u_\kappa + \sum_{\kappa=1}^{n} Q_\alpha^\kappa i_\kappa = 0 \qquad (\alpha=1, \cdots, n). \tag{2.6}$$

The system of equations (2.6) has many equivalent expressions. Especially, if we can choose a subset K of $\{1, \cdots, n\}$ such that (2.6) may be rewritten in the equivalent form:

$$\begin{aligned} i_\kappa &= \sum_{\lambda\in K} y_{\kappa\lambda} u_\lambda + \sum_{\lambda\notin K} h_{\kappa\lambda} i_\lambda \qquad (\kappa\in K), \\ u_\kappa &= \sum_{\lambda\in K} g_{\kappa\lambda} u_\lambda + \sum_{\lambda\notin K} z_{\kappa\lambda} i_\lambda \qquad (\kappa\notin K), \end{aligned} \tag{2.7}$$

then we say that the n-port has the hybrid immittance matrix:

$$\begin{bmatrix} Y & H \\ G & Z \end{bmatrix}. \tag{2.8}$$

For the existence of a hybrid immittance matrix, it is evidently necessary and sufficient that it is possible to choose a set of n linearly independent columns of $[P|Q]$ such that no pair of columns, one from the P-part and the other from the Q-part, with the same index κ may be chosen. Once we look at the problem with the matroid intersection problem in mind, we readily see that it suffices to consider the maximum-cardinality intersection problem between the first matroid $M_1 = (E, \rho_1)$ on the column set E of the matrix $[P|Q]$ where $\rho_1(X)$ $(X \subseteq E)$

is the number of linearly independent columns among X and the second matroid $M_2 = (E, \rho_2)$ on the same set E where $\rho_2(X)$ $(X \subseteq E)$ is the number of different κ-indices among X. Thus, we have an efficient algorithmic solution to Oono's problem. Furthermore, the principal-partition technique will reveal a fine structural properties of the n-port. In fact, the appearance of $F(\lambda)$'s with λ different from zero indicates the existence of singular ports, such as those which electrical engineers call "nullators" and "norators", and $F(0)$ affords useful information about how to enumerate all the hybrid immittance matrices.

This kind of formulation of Oono's problems is originally proposed by Iri and Tomizawa [25]. Recski treated a similar problem in a more general setting of "terminal solvability" [38a].

2.7. Controllability/observability of a linear dynamical system with combinatorial constraints [23]

A discrete-time linear dynamical system in R. Kalman's sense is defined by a system of difference equations of the following type:

$$\begin{aligned} \mathbf{x}(t+1) &= A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \\ \mathbf{y}(t) &= C\,\mathbf{x}(t) + D\,\mathbf{u}(t) \end{aligned} \qquad (t=0,1,2,\cdots), \qquad (2.9)$$

where $\mathbf{x}(t)$ is the n-dimensional state vector, $\mathbf{u}(t)$ the r-dimensional control vector, and $\mathbf{y}(t)$ the p-dimensional observation vector, respectively, at time t, and A, B, C and D are constant matrices of appropriate sizes.

It is the famous theorem due to Kalman that

the system is controllable, i.e. it can be brought to any state starting from an arbitrarily given state $\mathbf{x}(0)$ by choosing an appropriate sequence of controls $\mathbf{u}(0), \mathbf{u}(1), \cdots$, if and only if the rank of the wide matrix

$$[B, AB, A^2B, \cdots, A^{n-1}B] \qquad (2.10)$$

is equal to n,

and

the system is observable, i.e. the initial state $\mathbf{x}(0)$ (and, hence, all the subsequent states $\mathbf{x}(1)$, $\mathbf{x}(2)$, $\cdots$) can be identified based on the sequence of observations $\mathbf{y}(0)$, $\mathbf{y}(1)$, $\cdots$ (together with the information about the control sequence $\mathbf{u}(0)$, $\mathbf{u}(1)$, $\cdots$), if and only if the rank of the tall matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.11}$$

is equal to n.

This theorem relies substantially upon the assumption that no restriction is imposed upon the way of controlling or observing the system. However, in practical circumstances, the way of controlling and/or observing it would be subject to various restrictions. The typical smong them will be:

(i) every control terminal cannot be used more than a prescribed number of times, i.e. $u_i(t)$ may be different from zero at most a certain prescribed number of times for every i ;

(ii) at every time, at most a prescribed number of control terminals may be used, i.e. at most a prescribed number of $u_i(t)$'s may be different from zero for every t ;

(i') every observation terminal cannot be used more than a prescribed number of times, i.e. at most a prescribed number of $y_j(t)$'s for every j may be used in the estimator of the initial state;

(ii') at every time, at most a prescribed number of observation terminals may be used, i.e. at most a prescribed number of $y_j(t)$'s for every

t may be used in the estimator of the initial state.

In the case of controllability problem, if we set $E = \{(t, i) | i=1, \cdots, r; t = 0, 1, 2, \cdots\}$, make element (t, i) of E correspond to the i-th column of A^tB in the (infinitely) wide matrix

$$[B, AB, A^2B, \cdots] \qquad (2.12)$$

and consider the matroid (E, ρ) with the rank function ρ which, for any subset X of E, gives the number of linearly independent columns among X, then Kalman's theorem may be restated that the system is controllable if and only if the rank of the entire matroid is equal to n and that a base of the matroid can be chosen from the subset $\{(t, i) | i=1, \cdots, r; t=0, 1, 2, \cdots, n-1\}$. The combinatorial constraints such as (i) and (ii) define a family K $(\subseteq 2^E)$ of subsets of E such that each element of K corresponds to an usable set of pairs of control terminal and time. In entirely the same manner that we prove Kalman's theorem, we can prove that the system is controllable under the combinatorial constraints if and only if there is an element in $K \cap I$ whose cardinality is equal to n, where I is the family of independent sets of $M = (E, \rho)$. If the family K satisfies the axioms (I0)~(I2) in 1° of §1 --- this is indeed the case for consraints of types (i), (i'), (ii) and (ii') ---, then another matroid $M' = (E, \rho')$ can be defined on E, so that the controllability problem of the system is reduced to the maximum-cardinality intersection problem of two matroids.

The matroids here concerned, however, are apparently defined on a countably infinite set E. This might seem to invalidate the solution algorithm for finite matroids. Fortunately, it can actually be proved, for a large class of constraints including all the above examples, that, if $K \cap I$ has an element of cardinality n at all, then there is such one in $K \cap I \cap \{(t, i) | i=1, \cdots, r; t=0,1, \cdots, n-1 \text{ (or } 2n - 1)\}$, so that the problem is reduced to the intersection problem of two "finite" matroids.

2.8. Information theory [14], [15], [16], [16a]

The most fundamental concept in C. Shannon's information theory is the entropies of information sources, in terms of which the amount of "information" emanating from the sources can be measured quantitatively. If E is a set of a certain finite number of information sources, the entropy function h is defined as a mapping $h: 2^E \to R$, $h(X)$ being the entropy of a subset X of the sources. It is well recognized that h satisfies the conditions of

(i) nonnegativity: $0 \leqq h(X)$ for any X ($\subseteq E$),

(ii) monotonicity: $0 \leqq h(X) \leqq h(Y)$ for any X and Y such that $X \subseteq Y \subseteq E$

and

(iii) submodularity: $h(X \cup Y) + h(X \cap Y) \leqq h(X) + h(Y)$ for any X and Y ($\subseteq E$).

These three conditions are the same as the axioms which characterize the rank function of a polymatroid (see 1° of §1).

So, it might be expected that a fairly large part of the Shannon-type theory of information could be nicely rewritten in the language of matroids and polymatroids. In fact, S. Fujishige pointed out this relationship between the Shannon theory and polymatroids, and worked out a number of examples along these lines [14]. It will be interesting to investigate which part of traditional information theory depends only upon the polymatroidal structure of entropy functions and which part depends essentially upon the specific form --- $\sum p \log p$. T.-S. Han [15], [16], [16a] investigated in the same vein several problems such as that of encoding correlated information sources for multiple channels [4], [39] and of multiuser channels [34], [51].

When an information-theoretical problem is formulated in terms of polymatroids, the principal partition will enable us to recognize new concepts and new structure in communication systems. For example, Fujishige [14] showed that the principal partition classifies a set of correlated information sources according to the magnitude of conditional entropy (per source) of the sources of a class with respect to those of lower classes (lower in the partial order defined among the blocks of the partition), and gave, based on Edmonds' "greedy algorithm", a systematic method of encoding correlated information sources to be transmitted through separate channels. T.-S. Han [16a] treated this latter problem in more detail.

2.9. Scene analysis

Quite recently, K. Sugihara is applying the concept of submodular function, as well as the technique of principal partition associated with it, to a kind of so-called scene analysis. His problem is related to the degrees of over- and underdeterminacy of a configuration on the plane which is supposed to be the projection of a polyhedral complex (hidden lines being exlicitly shown in some cases and not in others). The reconstructability of the polyhedral complex from its projection is also a problem. These problems have been shown to be tractable combinatorially by the help of submodular functions defined in connection with the incidence relations among points, lines and faces of the projection [40a].

Similar techniques will apply to the analysis of the statical indeterminacy and the structural instability of elastic structures and link mechanisms, and to some problems in descriptive geometry.

2.10. Dual networks and inverse networks

This is an example not directly related to principal partition, but one for showing that reflecting upon fundamental concepts in some field of engineering science from the matroidal point of view will lead us to deeper understanding.

Reformulating and generalizing the fundamental concepts in electric network theory such as dual network, inverse network, adjoint network and reciprocal network, Iri and Recski recently pointed out that there are two "different" kinds of voltage-current symmetry [22a]. Usually, the duality in electric network theory is connected with the operation of interchanging the role of voltages and that of currents. This coincides with the mathematical duality in vector spaces, when we describe the performance of linear electric networks by linear equations, if --- and only if --- we deal with Kirchhoff's laws and/or reciprocal networks. In other words, the interchange of voltages and currents does not in general coincide with the mathematical duality. If we call the networks connected with each other by the former operation "inverse networks" and those connected with each other by the latter "dual networks", then we can define the "adjoint network" as the inverse of the dual under a very general condition, i.e. even for networks containing singular elements such as nullators and norators. By so doing, the mutual relations among the concepts mentioned in the above are made clear.

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The "IECEJ" stands for the "Institute of Electronics and Communication Engineers of Japan".

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