数理解析研究所講究録
第 397 巻 1980 年 234-248

ADVENTURES WITH RAMSEY THEORY

Frank HARARY, University of Michigan, Ann Arbor, MI, U.S.A.

Professor Iri has introduced me as the most distinguished mathematician in the world. This is not true. Maybe I am one of the best-known graph theorists but certainly not mathematician. I'll tell you one of my favorite introductions. I was in England and, as you know, the English don't always write things the way they pronounce them. They write Leicester, but they pronounce it Lester. The chairman of the Mathematics Department there stood up and said, "Today's speaker is so well known that he needs no introduction." And he sat down.

The following notation will be very useful, so it is introduced at once. Let $G \to F,H$ mean that, whenever you color the edges of G with two colors, without loss of generality green and red (like traffic lights--green means go; red means stop), you must get a green graph F or a red graph H. There is no avoiding it.

Now, where did this notation $G \to F,H$ come from? Paul Erdös, who is really the most active mathematician in the world, invented this use of the arrow and wrote $G \to G_1,G_2$. Then, David Sumner, without realizing it, just took the letters on both sides of letter G (standing for graph).

I was very happy to see my initials in Sumner's paper, because in 1962 I discovered the ramsey numbers for those graphs which are not necessarily complete.

Here is an example for the complete graphs, which is very well known: $K_6 \rightarrow K_3, K_3$. For brevity, $G \rightarrow F$, F is also written $G \rightarrow F$ when the two graphs F and H are the same, so here we can write $K_6 \rightarrow K_3$. This is usually stated as a puzzle problem as follows:

Consider six people at a party where some of them know each other and some do not know each other. Now, if they know each other, you draw a green line between the two points because they can talk to each other. If they don't know each other, you draw a red line. Now we will prove that $K_6 \rightarrow K_3$, meaning that, whenever you have six people at a party, there must be a triangle consisting of people who all know each other or a triangle of strangers, in other words--a green triangle or a red triangle. Here's how you prove it. First, we note the lemma that K_6 arrows $K_{1,3}$, the graph of the letter Y . To show this, consider person number 1. There are five other people, and five is an odd number. So, either he must know at least three people or not know at least three. Therefore, there must be a green $K_{1,3}$ or a red $K_{1,3}$ proving the lemma.

We now use this preliminary observation $K_6 \rightarrow K_{1,3}$ to show that $K_6 \rightarrow K_3$. Let's say that person 1 is a friendly chap, and he knows these three people. Now there are two possibilities: either two of these three people already know each other (in that case, you have a green triangle), or none of these three people know each other (then you have a red triangle).

The original theorem of Frank P. Ramsey, published in 1930 in the Journal of the London Mathematical Society, is that $K_\infty \to K_\infty$ where ∞ is the cardinality of the integers.

Now we are up to ramsey numbers. The graph theoretic corollary of Ramsey's original theorem is that for all m and n there is a number p such that $K_p \to K_m, K_n$. (If you are a number theorist, p is always a prime number. I am not a number theorist, so I use p for the number of points. And if you are a group theorist, G is always a group. But I am not a group theorist, so for me G is always a graph.) Then you can define the classical ramsey number $r(m,n)$ as the minimum p: K_p arrows K_m, K_n . (The colon, : , means "such that", the top dot is SUCH and the bottom one is THAT.) As a corollary to this corollary of Ramsey's theorem, for all graphs F and H with no isolated points, there exists a number p: K_p arrows F,H . To prove this, let F have m points and H have n points. Since there is a p: K_p arrows K_m, K_n , the green K_m (if any) must contain F or the red K_n contains H.

The classical ramsey numbers $r(2,n) = n$ are really trivial to verify. For on n points, if there is no green K_2 there must be a red K_n . Hence, we only need consider $r(m,n)$ for $3 \leq m \leq n$ as obviously $r(m,n) = r(n,m)$. In 1969, my book Graph Theory, listed exactly six classical ramsey numbers which were determined exactly, as in Table 1. Now it is a decade later, and no other classical ramsey numbers have since been calculated. This appears to be a very difficult problem. There are no powerful methods known, only exhaustive consideration of all possible cases. Up to now, computer programs have not been helpful either.

TABLE 1. All the known ramsey numbers $r(m,n)$ with $3 \leq m \leq n$.

m \ n	3	4	5	6	7
3	6	9	14	18	23
4		18			

Twice now in my career, I have been a Visiting Professor--once for one year in London, once for a half year in Waterloo; both times Erdös was a Visiting Professor in the same university at the same time! This was very lucky for me--partly because Erdös is the one who taught me how to travel, but mostly because he is such a great and curious mathematician whose enthusiasm is contagious. In fact, it was in my book, A Seminar on Graph Theory, that I first began to write these little stories, which I hope will eventually come out as a somewhat autobiographical book printed in Japanese with Professor Akiyama translating it. Erdös talks about mathematics to everyone and writes joint papers with almost everyone--I also learned that from him. He was once in a train in Hungary, and the friendly conductor said, "Old man, you look like an interesting person. Are you a scientist?" He replied, "I am a mathematician. Let me tell you my latest research." And he began to tell him new theorems. Later, a rumor circulated that he wrote a joint paper with the conductor. When I asked him if it was true, he said, "No. I wrote it myself."

In London, after hearing Erdös speak on $r(m,n) = r(K_m,K_n)$, I told him that you can define $r(F,H)$ for any two graphs F and H with no isolates, not necessarily complete graphs. He replied that it will not be interesting. And I said "It will!" He said "It won't!" Again I said "It will!" and he insisted "It won't!", so I did not work on it then.

Eight years later in Waterloo, Ontario, Canada, he and I were having a cup of tea. Suddenly, he said, "You must come with me right now! There is a very good show about to begin. A genius is defending his thesis, and his name is Václav Chvátal." His thesis was on ramsey theory for hypergraphs; and I could see he was so smart that, if I told him my idea about the ramsey number of not necessarily complete graphs, he will understand it at once. I offered to share it with him, because I am in the happy situation that I have more research ideas than the time to write all of them down, so I share them with people. Last year, I was very lucky to have Professor Akiyama to share ideas with in Ann Arbor; and he also had many original ideas and shared them with me.

When I shared this "generalized ramsey theory" with Chvátal, it led to four joint papers. First, we sent a research announcement to the Bulletin of the American Mathematical Society. Then in Ramsey I, the first numbered paper in my series, there were some ramsey numbers given for stars and there were some bounds. We sent this one to Periodica Mathematica Hungarica. Because it was a new journal, we thought it would be published quickly. In fact, the second paper came out before the first one did!

As a joke, we defined a "small graph" as one with no isolated points and with $p \leq 4$ points. There are just ten small graphs, as shown in Figure 1 together with the conventional notation for graphs as given in my book, Graph Theory (=GT). By the way, in 1979, more copies of the Japanese translation of GT were sold in Japan than copies in English all over the world. Thus, eventually, Japan may become Number 1 in Graph Theory research just as she is already almost Number 1 in many branches of industry. It is basic research which leads to progress in engineering and industry, and it is research in graph theory which assists both basic and applied research because it is so very useful as an appropriate mathematical model.

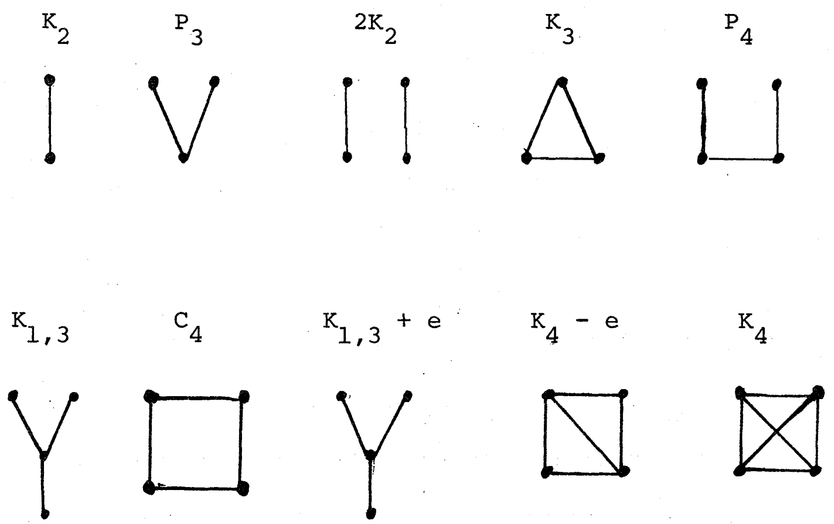

Figure 1. The ten small graphs and their names

In Ramsey II, we found the small diagonal numbers $r(F)$, and in Ramsey III, the off-diagonal numbers, $r(F,H)$. These numbers are all given in Table 2 using the notation from Figure 1.

TABLE 2. The diagonal and off-diagonal ramsey numbers of the small graphs.

	K_2	P_3	$2K_2$	K_3	P_4	$K_{1,3}$	C_4	$K_{1,3}$+e	K_4-e	K_4
K_2	2	3	4	3	4	4	4	4	4	4
P_3		3	4	5	4	5	4	5	5	7
$2K_2$			5	5	5	5	5	5	5	6
K_3			6	7	7	7	7	7	7	9
P_4					5	5	5	7	7	10
$K_{1,3}$						6	6	7	7	10
C_4							6	7	7	10
$K_{1,3+e}$								7	7	10
K_4-e									10	11
K_4										18

When Chvátal got a job as an assistant professor at Stanford, he was too far from Ann Arbor to talk with except for an occasional telephone call. One day I was having lunch with my Doctoral Son Number 2, Geert Prins, in Detroit in a Chinese restaurant. He said, "Frank, I have more time than ideas, and you have more ideas than time. Please give me a problem to work on." And so I told him my definition of the ramsey multiplicity of a graph F, denoted by $R(F)$ to distinguish it from the ramsey number $r(F)$. By $R(F)$ is meant the smallest possible number of monochromatic (all one color, either green or red) copies of F which can occur in any 2-coloring of the lines of the complete graph having $r(F)$ points.

The way I invented ramsey multiplicity was that I noticed that there was a well-known little theorem that when you take six points and color each line of K_6 either green or red, you not only must get one triangle all one color, but you must get at least 2! So, I said the ramsey multiplicity of K_3 is 2 , written $R(K_3) = 2$. What are the ramsey multiplicities of all other small graphs? Obviously $R(K_2) = 1$ for as soon as you color the one line of K_2 green or red, you have made just one copy of K_2 , and similarly $R(P_3) = 1$. It is immediate that $R(2K_2) = 3$, as $r(2K_2) = 5$ and the lines of K_5 can be colored with green $K_3 \cup K_2$ and red $K_{2,3}$ giving three green copies of $2K_2$. Also, I had already verified by exhaustion that $R(C_4) = 2$. Then Prins found the ramsey multiplicity for all the other small graphs except K_4 ; that's still an unsolved problem! In particular, the fact that the ramsey multiplicity of the random graph $K_4 - e$ is 15 was found with the help of Allen Schwenk (Doctor Number 12). Thus, we thought the ramsey multiplicity of K_4 would be tremendous, but an Englishman recently proved that it is at most 12 and said that he thinks 12 will be the exact value. I am inclined to agree with his conjecture. Table 3 shows the ramsey multiplicity of the small graphs.

TABLE 3. The ramsey multiplicity of the small graphs

F	K_2	P_3	$2K_2$	K_3	P_4	$K_{1,3}$	C_4	$K_{1,3}+e$	K_4-e	K_4
r(F)	2	3	5	6	5	6	6	7	10	18
R(F)	1	1	3	2	10	3	2	12	15	12?

The ramsey multiplicity of a graph was only just introduced in our 1974 paper and already there is going to be a survey of the subject by Burr and Rosta, which is to appear in the only journal entirely devoted to our beautiful subject, the Journal of Graph Theory.

Pavol Hell is Chvátal's best friend; they were fellow students in Prague, Czechoslovakia. Pavol and I were in Las Vegas at the same time, not to gamble, but to attend a meeting of the American Mathematical Society. Hell and I were walking along "the Strip," which is what they call the main street in Las Vegas; and he said, "You write papers with so many people, let's write a paper together." I said, "Okay, let's study the ramsey number of a directed graph." We agreed and discussed it while we walked for an hour on the Strip. Later, he wrote out the rest of it and sent it to me, and I rewrote it and so forth and that became Ramsey V. And now here comes a coincidence that happens so often in graph theory. A French gentleman called Bermond in Paris wrote a paper with almost exactly the same results, and it appeared the same month. Independent discovery happens so often! Most of the time, with independent discoveries, I am the winner since I fell in love with graph theory in 1950 which was earlier than most people now fascinated by graphs. So when the same result is discovered independently, I usually found it first. But I have been the loser. In fact, Bob Norman (Doctor Number 1) and I derived some equations about matchings and coverings; and we submitted them to Paul Halmos for possible publication in the Proceedings of the American Mathematical Society. By return mail, we received word that Gallai had done it ten years earlier, so we were the losers. Sometimes you win; sometimes you lose.

The next one was with Dick Duke on simplicial complexes, because we were both interested in this. I was visiting him in Seattle in his house boat. Duke University in North Carolina was founded by somebody named Duke with millions of dollars made from manufacturing cigarettes. There they have the Duke Mathematical Journal. So I thought it would be a great joke if I sent the paper Ramsey VI with Richard Duke to the Duke Mathematical Journal. Unfortunately, they sent back the paper by return mail, not refereed, with a letter saying that their backlog is over three years, so they will not consider any more paper this year. It was a great pity. Instead, it went to the Journal of the Australian Mathematical Society.

When Allen Schwenk was my doctoral student, he was the one that I shared all my ideas with because he was smart and HE WAS THERE. So, I gave him the idea of finding out whether or not there is a ramsey number for multigraphs and for networks. If you have a network, the value on a line might be 0.7. Then, maybe you color 0.4 of it green and 0.3 of it red, and you see what you can get. Schwenk proved a very nice theorem that either the ramsey number for a given multigraph does not exist or it is equal to the ramsey number of its underlying graph. It's a nice theorem, and because it was subtitled Multigraphs and Networks, Ramsey VII was published in the journal, Networks.

Zevi Miller (Doctor Number 13) developed my idea which turn out to be an independent discovery again. Let $\hat{r}(F,H)$, called in English "r-hat," be the smallest number of lines in a graph G such that $G \rightarrow F,H$. This is called the size ramsey number because recently some graph theorists are calling the number p of points in G its order; and the number q of the lines the size. For example, we found that the size ramsey number of the random graph is 44. You see, its ramsey number is 10, and the number of lines in K_{10} is 45. We found that when you take any one line out of K_{10}, the resulting graph still arrows the random graph; but if you remove any two lines, it does not! We found the size ramsey number for all the small graphs except K_4. This is still an unsolved problem.

Now another independent discovery! At the same time in Memphis, Tennessee, the two ramsey specialists Faudree and Shelp were also discovering the size ramsey number! They wrote a paper on it jointly with their colleague Rousseau and with the great Erdös himself, but it did not overlap ours as they found bounds.

Ramsey IX was with Robert Robinson. We were in Canberra, Australia for the two weeks beginning 14 August 1977 at a combinatorial meeting. There Robinson and I developed the isomorphic ramsey number of a graph F. Here you have to color the lines of a complete graph in such a way that the two parts (one green and one red) are isomorphic, and you get a self-complementary graph. Or, if you use three colors, you are going to have three isomorphic graphs: Green, Red, Blue, and you will have isomorphic ramsey numbers of three colors, etc. This invariant is a combination of ramsey theory with isomorphic factorization. What started us on this was that we noticed that every self-complementary graph with 8 points contains a triangle, not with 5 points, and there aren't any such graphs with 6 or 7 points. Thus the isomorphic ramsey number of a triangle is 8, and we found the isomorphic ramsey numbers for all the small graphs.

Ramsey X was with Grossman and Klawe. Grossman is at Oakland University, 60 miles from Ann Arbor, and Maria Klawe was also teaching there then. She came to a lecture that I gave in Windsor, Onatario, just across the bridge from Detroit, because she had never seen me and had heard a rumor that sometimes I give an interesting talk. She asked me if I would please give her a question to work on. Of course, I said Okay: find the ramsey numbers for all the double stars. She talked it over with Grossman, and they worked together; so they both joined me, which was very agreeable. By the way, we dedicated this paper to Ron Graham and Paul Erdös calling them double stars! They really are stars in mathematics because they shine and sparkle, and are adjacent to so many other mathematicians

Now we are up to Ramsey XI! The reason for the exclamation point is that is is the first article in my ramsey series which I wrote by myself. I was asked the other day which theorem that I found do I like best? This is my favorite personal theorem. A positive integer n is a ramsey number if and only if n is different from 4 . Four is the only number which is not the ramsey number for some graph. So, if you want to make a joke, you could say that this is a new way to define the number 4. The proof is not immediate, but it's easy after you know it. I'll just describe it to you. It is clear that 4 cannot be a ramsey number because the two graphs having two lines have ramsey numbers 3 and 5 (Table 2), and the smallest ramsey number of the graphs with three lines is 5. When you have more lines, you can never get a lower ramsey number; so 4 <u>is not</u> a ramsey number. To show that every $n \neq 4$ <u>is</u> a ramsey number, I looked at the known formulas for the ramsey numbers of the paths, the stars, the cycles, and so forth. Putting them all together, I got every number as a ramsey number except for n = 16. Then Grossman told me that we already have 16 and he showed me our paper, Ramsey X, where the double star S(4,6) has ramsey number 16. It was already in our paper, but I had not noticed it. It was like a detective story.

The Ramsey XII paper with Harborth and Mengersen came to exist because I gave a lecture in Braunschweig, Germany. We studied bipartite ramsey sets. This means you start with a complete bipartite graph and color its lines green or red. Then for a given bipartite graph F , a pair of numbers (m,n) is in the bipartite ramsey set of F if m and n are minimal for guaranteeing that the bipartite graph F will be formed in one color, i.e., that $K_{m,n} \rightarrow F$.

Finally, Ramsey XIII, also written by myself, is to appear in the proceedings book of the Fourth International Conference on Graph Theory. It was held in Kalamazoo, Michigan during 6 - 9 May 1980 and was dedicated to both my friend Edward Nordhaus, on the occasion of his retirement from Michigan State University,and to me to celebrate in advance my forthcoming Sixtieth Birthday which is scheduled to take place on 11 March 1981. My talk was both a technical summary of the previous papers in this ramsey series and introduced achievement and avoidance numbers for graphs.

I take this opportunity to express my heartfelt thanks to all my co-authors of these ramsey papers. It has been a pleasure for me to do research with such brillant and dedicated scholars, who are now listed in alphabetical order:

Chvátal, Václav

Duke, Richard

Grossman, Jerrold

Harborth, Heiko

Hell, Pavol

Klawe, Maria

Mengersen, Ingrid

Miller, Zevi

Prins, Geert

Robinson, Robert

Schwenk, Allen

It gives me special pleasure to announce that two additional papers in this series are being written with a world expert on matroid theory, Masao Iri. One will develop the reduced ramsy number of a graph by combining homomorphisms with subcontractions; the other will introduce the ramsey number of a uniform matroid.

REFERENCES

0. V. Chvátal and F. Harary, Generalized ramsey theory for graphs. Bull. Amer. Math. Soc. 78 (1972) 423-426.

1. V. Chvátal and F. Harary, Generalized ramsey theory for graphs I: Diagonal numbers. Periodica Math. Hungar. 3 (1973) 115-124.

2. V. Chvátal and F. Harary, Generalized ramsey theory for graphs II: Small diagonal numbers. Proc. Amer. Math. Soc. 32 (1972) 389-394.

3. V. Chvátal and F. Harary, Generalized ramsey theory for graphs III: Small off-diagonal numbers. Pacific J. Math. 41 (1972) 335-345.

4. F. Harary and G. Prins, Generalized ramsey theory for graphs IV: The ramsey multiplicity of a graph. Networks 4 (1974) 163-173.

5. F. Harary and P. Hell, Generalized ramsey theory for graphs V: The ramsey number of a digraph. Bull. London Math. Soc. 6 (1974) 175-182; 7 (1975) 87-88.

6. R.M. Duke and F. Harary, Generalized ramsey theory VI: Ramsey numbers for small plexes. J. Austral. Math. Soc. 29 (1976) 400-410.

7. F. Harary and A.J. Schwenk, Generalized ramsey theory VII: Multigraphs and networks. Networks 8 (1978) 209-216.

8. F. Harary and Z. Miller Generalized ramsey theory for graphs VIII: The size ramsey number. Studies in Pure Mathematics in Memory of Paul Turán (1981), to appear.

9. F. Harary and R.W. Robinson, Generalized ramsey theory for graphs IX: Isomorphic factorizations IV: Isomorphic ramsey numbers. Pacific J. Math. To appear.

10. J. Grossman, F. Harary, and M. Klawe, Generalized ramsey theory for graphs X: Double stars. Discrete Math.

11. F. Harary, Generalized ramsey theory for graphs XI: Which numbers are graphical? Nanta Math. 12 (1979) 117-124.

12. F. Harary, H. Harborth and I. Mengersen, Generalized ramsey theory for graphs XII: Bipartite ramsey sets. J. Glasgow Math. Soc. (1981) to appear.

13. F. Harary, Generalized ramsey theory for graphs XIII: Achievement and avoidance numbers. Proceedings Fourth International Conference on Graph Theory New York, Wiley (1981) to appear.

14. F. Harary, A Seminar on Graph Theory. Holt, New York (1967).

15. F. Harary, Graph Theory. Addison-Wesley, Reading (1969).

16. F.P. Ramsey, On a problem of formal logic. Proc.London Math. Soc. 30 (1930) 264-286.