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On ideal-adic completion of noetherian rings and its application

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0. Introduction.

In commutative (noetherian) ring theory, complete local rings have a lot of good properties and play many important roles to study the (local) properties of (general) noetherian rings.

Nagata used reduced complete local rings in the investigation, for example, of the finiteness problem of integral closures of noetherian domains. In this work, he found a good class of noetherian rings which possess the (universal) finiteness property for integral closures and named these rings "pseudo-geometric" (= universally japanese). We should note that he produced many examples of bad local domains at the same time.

In reconstructing Nagata's work, Grothendieck noticed the importance of the informations included in formal fibres, which connect a local ring with its completion. He paid a special attention to the study of noetherian rings whose formal fibres are geometrically regular. Then, he found a new class of noetherian rings which have algebraic- geometrically reasonable properties and called them "excellent".

Since complete local rings are proved to be always excellent, Grothendieck expected that the conditions of formal fibres might become no worse when one completes a noetherian ring in any ideal-adic topology. More precisely, letting $\mathbb{P}$ denote a certain (ring-theoretic) property, Grothendieck defined a $\mathbb{P}$-ring to be a noetherian ring which has the property $\mathbb{P}$. In these terminologies, he asked the following questions (cf. [2, (7.4.8)]):

Question 1. Let A be a noetherian ring and I an ideal of A . If A is a $\mathbb{P}$-ring, is the I-adic completion A* of A also a $\mathbb{P}$-ring ?

Question 2. With A and I as above, suppose

a) A is complete and separated in the I-adic topology, and

b) A/I is a $\mathbb{P}$-ring.

Does it follow that A is also a $\mathbb{P}$-ring ?

In this note, we study the above questions in the case where $\mathbb{P}$ = universally japanese, a G-ring, a Z-ring or an N-ring.

Marot [5] showed that, if $\mathbb{P}$ = universally japanese, then Question 2 is always true. When A is a semi-local ring and $\mathbb{P}$ = a G-ring (or a Z-ring, an N-ring), Rotthaus [14] proved that Question 2 is also valid in these cases.

On the other hand, when A is a (general) noetherian ring and $\mathbb{P}$ = a G-ring (or a Z-ring, an N-ring), an elementary example shows that even Question 1 is not true.

In this paper, we mainly use the notations and the definitions in [2], [7], [8].

1. Mori-Nagata Theorem.

In this section, we give an elementary proof of Mori-Nagata Theorem on integral closures of noetherian domains. Note that the original proof of Mori-Nagata Theorem is based on the structure theorem of complete local rings. Our proof essentially uses henselization instead of completion.

We first give a generalization of Krull-Akizuki Theorem, due to Matijevic:

(1.1) Theorem. (Matijevic [6]) If A is a noetherian ring and $T(A)$ is the global transform of A, then for any ring B such that $A \subset B \subset T(A)$, B/xB is a finite A-module for each non-zero-divisor x in A. In particular, if A is reduced, B is always noetherian.

(1.2) Corollary. (Krull-Akizuki Theorem) Let A be a noetherian domain with field of quotients K, L a finite algebraic extension of K and B a ring such that $A \subset B \subset L$. If $\dim A = 1$, then

(1.2.1) B is a noetherian ring of dimension at most one, (1.2.2) for a prime ideal P of A , there are only a finite number of prime ideals Q of B such that $Q \cap A = P$, and

(1.2.3) $[k(Q) : k(P)]$ is finite.

(1.3) Theorem. (Mori-Nagata [8, (33.10)]) Let A be a noetherian domain and $\bar{A}$ the derived normal ring of A . Then

(1.3.1) $\bar{A}$ is a Krull domain,

(1.3.2) for a prime ideal P of A , there are only a finite number of prime ideals $\bar{P}$ of $\bar{A}$ such that $\bar{P} \cap A = P$, and

(1.3.3) $[k(\bar{P}) : k(P)]$ is finite.

In the proof of Mori-Nagata Theorem, we use several properties of Krull domains. We refer the reader to [3, §1].

We first show (1.3.3) for henselian local domains by induction on dimension, starting with Krull-Akizuki Theorem:

(1.4) Proposition. Let (A , m) be a henselian local domain with field of quotients K , L a finite algebraic extension of K and (B , n) the integral closure of A in L . Then $[B/n : A/m]$ is finite.

(1.5) (cf. [8, (43.20)], [13, Chapitre IX]) (1.3.2) and (1.3.3) for general noetherian domains are proved by (1.4) and the following canonical correspondences:

Let (A, m) be a noetherian domain with field of quotients K, $\overline{A}$ the derived normal ring and $({}^hA, {}^hm)$ the henselization of A. Then we have three natural one-to-one correspondences between:

(1.5.1) the maximal ideals $\{\overline{m}_i\}$ of $\overline{A}$ and those $\{m_i^*\}$ of ${}^hA \otimes_A \overline{A}$,

(1.5.2) the maximal ideals $\{m_i^*\}$ of ${}^hA \otimes_A \overline{A}$ and the minimal prime ideals $\{q_i^*\}$ of ${}^hA \otimes_A \overline{A}$, and

(1.5.3) the minimal prime ideals $\{q_i^*\}$ of ${}^hA \otimes_A \overline{A}$ and those $\{q_i\}$ of ${}^hA$.

In these correspondences,

(1.5.4) $\overline{A}/\overline{m}_i = ({}^hA \otimes_A \overline{A})/m_i^*$,

(1.5.5) $({}^hA \otimes_A \overline{A})_{m_i^*}$ is the derived normal ring of ${}^hA/q_i$, and

(1.5.6) $({}^hA \otimes_A \overline{A}) \cap K = \overline{A}$.

Before proving (1.3.1), we remark:

(1.6) Proposition. ([11, (1.7)]) Let (A, m) be a local domain and $m = a_1A + \ldots + a_rA$. Then the total transform $A(m)$ $(= \bigcap_{i=1}^{r} A_{a_i})$ of A is integral over A if and only if the derived normal ring has no maximal ideal of height one. In particular,

(1.6.1) if $\overline{A}$ has a maximal ideal of height one, then $\text{prof}\, A = 1$.

(1.7) Corollary. Let (A, m) be a henselian local domain of dimension greater than one, $m = a_1A + \dots + a_rA$ and $\overline{A}$ the derived normal ring of A. Then $\overline{A} = \bigcap_{i=1}^{r} \overline{A}_{a_i}$.

Outline of the proof of (1.3.1) (cf. [10, Proposition 6]):
We prove (1.3.1) in a few steps.
Step 1. (Krull-Akizuki Theorem) (1.3.1) is true for all noetherian domains of dimension at most one.
Step 2. We have the following implications: Let n be a positive integer, [(1.3.1) is true for all noetherian domains of dimension at most n] $\Rightarrow$ [(1.3.1) is true for all henselian local domains of dimension at most $(n + 1)$] (cf. (1.7)) $\Rightarrow$ [(1.3.1) is true for all local domains of dimension at most $(n + 1)$] (cf. (1.5.6)) $\Rightarrow$ [(1.3.1) is true for all noetherian domains of dimension at most $(n + 1)$] (cf. (1.3.2), (1.6.1)).
Step 3. By induction on dimension, Step 2 shows that (1.3.1) is true for all local domains. Therefore, (1.3.1) is true for all noetherian domains (cf. (1.3.2), (1.6.1)).

2. Japanese Rings.

In this section, we give a slight generalization of a lemma of Tate. Then we show that some well-known theorems are derived from this.

First we note an easy lemma, which seems to be useful:

(2.1) Lemma. ([9]) Let A be a Krull domain and P a prime ideal of height one. If A/P is noetherian, then $A/P^{(e)}$ is noetherian for any natural number e .

(2.2) Corollary. Let A be a Krull domain. If A/P is noetherian for every prime ideal P of height one, then A is also noetherian.

(2.3) Corollary. (Mori) Let A be a noetherian domain and $\overline{A}$ the derived normal ring of A . Then $\overline{A}$ is noetherian if and only if $\overline{A}/\overline{P}$ is noetherian for every prime ideal $\overline{P}$ of height one.

(2.4) Corollary. (Mori-Nagata) The derived normal ring $\overline{A}$ of a noetherian domain of dimension two is again noetherian.

(2.5) Corollary. (Heinzer) Let A be a noetherian domain of dimension at most two and B a Krull domain such that $A \subset B$. If [Q(B) : Q(A)] is finite, then B is noetherian and of dimension at most two.

(2.6) Remarks. (cf. [8, Appendix])

(2.6.1) There exists a two-dimensional local domain A which has non-noetherian over-rings between A and its derived normal ring (cf. Example (4.5)).

(2.6.2) There exists a three-dimensional local domain A whose derived normal ring is not noetherian (cf. Example (4.8)).

Now we prove:

(2.7) Theorem. Let A be a noetherian domain and x an element of A. Suppose

(2.7.1) A is complete and separated in the xA-adic topology, and

(2.7.2) A/P is a japanese ring for every P in $\mathrm{Ass}(A/xA)$.

Then A is also a japanese ring.

Proof. Let L be a finite algebraic extension of $Q(A)$ and B the integral closure of A in L. Then B is a Krull domain (cf. (1.3.1)) and $xB = Q_1^{(e_1)} \cap \ldots \cap Q_s^{(e_s)}$ with height-one prime ideal Q_i of B. Let $Q_i \cap A = P_i$, then

(2.7.3) $P_i \in \mathrm{Ass}(A/xA)$ and $[k(Q_i) : k(P_i)]$ is finite for any i (cf. (1.6.1), (1.3.3)).

Since A/P_i is a japanese ring by assumption, B/Q_i is a finite (A/P_i)-module. Hence B/Q_i is noetherian. Thus, B/xB is noetherian by Lemma (2.1). Moreover, these imply that B/xB is a finite A-module. Therefore, as a Krull domain B is xB-adic separated and A is xA-adic complete, B itself becomes a finite A-module.

(2.8) Corollary. (Tate) Let A be a normal domain and x an element of A such that xA is a prime ideal. Suppose

(2.8.1) A is complete and separated in the xA-adic topology, and

(2.8.2) A/xA is a japanese ring.

Then A is also a japanese ring.

(2.9) Corollary. (Marot's Theorem [5]) Let A be a noetherian ring and I an ideal of A . Suppose

(2.9.1) A is complete and separated in the I-adic topology, and

(2.9.2) A/I is an universally japanese ring.

Then A is also an universally japanese ring.

(2.10) Corollary. (Nagata) A complete semi-local ring is an universally japanese ring.

3. Ideal-adic Completion of Semi-local $\mathbb{P}$-rings.

In this section, we define three classes of noetherian rings with good formal fibres. Then, we show that Question 2 has a positive answer for semi-local rings in these classes.

(3.1) Definition. Let A and B be noetherian rings and ψ a ring homomorphism of A to B . Then we call ψ regular (or normal, reduced) if

(3.1.1) ψ is flat, and

(3.1.2) for every prime ideal P of A , the induced homomorphism $\psi \otimes k(P)$ makes $B \otimes_A k(P)$ geometrically regular (or geom. normal, geom. reduced, resp.) over $k(P)$.

(3.2) Definition. Let A be a noetherian ring. Then we say A is a G-ring (or a Z-ring, an N-ring) if, for every maximal ideal m of A, the canonical homomorphism ρ_m of A_m to $(A_m)^\wedge$ is regular (or normal, reduced, resp.).

In these terminologies, we state the main theorem of this section:

(3.3) Theorem. (Rotthaus [14], cf. [5], [12]) Let A be a semi-local ring and I an ideal of A. Suppose

(3.3.1) A is complete and separated in the I-adic topology, and

(3.3.2) A/I is a G-ring (or a Z-ring, an N-ring).

Then A is also a G-ring (or a Z-ring, an N-ring, resp.).

To prove Rotthaus' Theorem, we need the following:

(3.4) Theorem. (André [1], cf. [12, Proposition (2.4)]) Let (A, m) and (B, n) be local rings and ψ a local homomorphism of A to B. Suppose

(3.4.1) ψ is flat,

(3.4.2) $\bar{\psi}$ $(= \psi \otimes k(m))$ is regular (or normal, reduced), and

(3.4.3) A is a G-ring (or a Z-ring, an N-ring, resp.).

Then ψ is also regular (or normal, reduced, resp.).

(3.5) Lemma. (Rotthaus [14, Lemma 2], cf. [12, Lemma (1.2)]) Let A be a noetherian ring and I an ideal of A. Let B be an A-algebra, b an ideal of B . Put $a = b \cap A$, $b_n = b + I^n B$ and $a_n = b_n \cap A$. Suppose

(3.5.1) A is complete and separated in the I-adic topology,

(3.5.2) $b_n = a_n B$ for any $n > 0$,

(3.5.3) $B/a_n B$ is faithfully flat over A/a_n for any $n > 0$, and

(3.5.4) $\bigcap_n b_n = b$ and $\bigcap_n (aB + I^n B) = aB$.

Then $aB = b$.

Proof of Rotthaus' Theorem. Since a complete (semi-) local ring is a G-ring (cf. [7, (30.D), Theorem 68]), by induction on $\dim A/I$, we may assume

(3.3.3) for any non-maximal prime ideal p of A which contains I , the I_p-adic completion A^*_p of A_p is a G-ring (or a Z-ring, an N-ring, resp.).

Moreover, by noetherian induction, we may assume

(3.3.4) A is a semi-local domain with field of quotients K and, for any non-zero ideal a of A , A/a is a G-ring (or a Z-ring, an N-ring, resp.).

Since any finite A-algebra satisfies the same assumptions as above, to prove the theorem, it is sufficient to show that $\hat{A} \otimes_A K$ is regular (or normal, reduced, resp.).

It is known that, if a semi-local ring R is a G-ring (or an N-ring, resp.), then the set $\mathrm{Reg}(R) = \{ P \in \mathrm{Spec}(R) \mid R_P \text{ is regular} \}$

(or $\mathrm{Nor}(R) = \{ P \in \mathrm{Spec}(R) \mid R_P$ is normal $\}$, resp.) is open in $\mathrm{Spec}(R)$ (in Zariski topology) (cf. [7, (32.A), (32.C), (33.D)]).

Hence, as $\hat{A}$ is a G-ring, we denote by $b = \mathrm{sing}(\hat{A})$ (or $b = \text{non-nor}(\hat{A})$, $b = \mathrm{nil}(\hat{A})$) the ideal of $\hat{A}$ which defines the (reduced) closed set $\mathrm{Sing}(\hat{A}) = \mathrm{Spec}(\hat{A}) - \mathrm{Reg}(\hat{A})$ (or $\text{non-Nor}(\hat{A}) = \mathrm{Spec}(\hat{A}) - \mathrm{Nor}(\hat{A})$, $\mathrm{Spec}(\hat{A})_{red}$, resp.).

Put $b_n = b + I^n\hat{A}$ and $a_n = b_n \cap A$. We claim:

(3.3.5) $a_n\hat{A} = b_n$ for any $n > 0$.

<u>Proof of</u> (3.3.5). Let $b_n = Q_1 \cap \ldots \cap Q_s$ be a primary decomposition of b_n , where Q_i is a P_i-primary ideal. Then, letting $p_i = P_i \cap A$ and $q_i = Q_i \cap A$ (a p_i-primary ideal), we have $a_n = q_1 \cap \ldots \cap q_s$ (, which may not be an irredundant decomposition). First we can easily check:

(3.3.5.1) if P_i is a maximal ideal of $\hat{A}$, then $q_i\hat{A} = Q_i$.

Hence, to get the claim, it is sufficient to show:

(3.3.5.2) if p is a non-maximal prime ideal of A which contains I , then $a_n\hat{A}_T = (b_n)_T$ with $T = A - p$.

<u>Proof of</u> (3.3.5.2). Let A_T^* be the I_T-adic completion of A_T and $(\hat{A}_T)^*$ the $I\hat{A}_T$-adic completion of $\hat{A}_T$. Let ρ (or τ) be the canonical homomorphism of A to $\hat{A}$ (or of $\hat{A}_T$ to $(\hat{A}_T)^*$, resp.) and let ρ_T (or ρ_T^*) be the induced homomorphism of A_T to $\hat{A}_T$ (or of A_T^* to $(\hat{A}_T)^*$, resp.). Then we have the following commu-

tative diagram:

$$\begin{array}{ccccc} \hat{A} & \longrightarrow & \hat{A}_T & \xrightarrow{\tau} & (\hat{A}_T)^* \\ \rho \uparrow & & \uparrow \rho_T & & \uparrow \rho_T^* \\ A & \longrightarrow & A_T & \longrightarrow & A_T^* \end{array}$$

By assumption (3.3.3), A_T^* is a G-ring (or a Z-ring, an N-ring, resp.). Then, as $\overline{\rho}_T^*$ ($= \rho_T^* \otimes (A_T^*/IA_T^*) = \rho \otimes (A/I)_T$) is regular (or normal, reduced, resp.) by assumption (3.3.2), ρ_T^* is also regular (or normal, reduced, resp.) by André's Theorem.

Let $c = \operatorname{sing}(A_T^*)$ (or $c = \text{non-nor}(A_T^*)$, $c = \operatorname{nil}(A_T^*)$, resp.). Then, as ρ_T^* is regular (or normal, reduced, resp.) and τ is regular, we have $c(\hat{A}_T)^* = b(\hat{A}_T)^*$. Consequently, letting $c_m = c + I^m A_T^*$,

(3.3.5.3) $c_m(\hat{A}_T)^* = b_m(\hat{A}_T)^*$ for any $m > 0$, hence

(3.3.5.4) $(a_n)_T = c_n \cap A_T$.

Conversely, as a_n contains I^n , (3.3.5.4) implies

(3.3.5.5) $a_n A_T^* = c_n$.

Then, by (3.3.5.3), we have

(3.3.5.6) $a_n(\hat{A}_T)^* = b_n(\hat{A}_T)^*$.

Therefore, as $a_n\hat{A}_T$ contains $I^n\hat{A}_T$, (3.3.5.6) implies $a_n\hat{A}_T = (b_n)_T$.

Thus (3.3.5.2) is proved and this also completes the proof of (3.3.5).

<u>Final step of the proof of Rotthaus' Theorem</u>. (With notation as above) By (3.3.5) and Lemma (3.5), we have $b = (b \cap A)\hat{A}$. Hence, as b is semi-prime, $({}^a\rho)^{-1}(\mathrm{Reg}(A)) = \mathrm{Reg}(\hat{A})$ (or $({}^a\rho)^{-1}(\mathrm{Nor}(A)) = \mathrm{Nor}(\hat{A})$, $\mathrm{nil}(A)\hat{A} = \mathrm{nil}(\hat{A})$, resp.). Therefore, $\hat{A} \otimes_A K$ is regular (or normal, reduced, resp.).

q.e.d.

4. Examples.

In this section, we first present an example which gives a negative answer to Question 1 (, where $\mathbb{P}$ = a G-ring, a Z-ring or an N-ring). We note that this example gives a new one of two-dimensional local domain which has non-noetherian over-rings between the domain and its derived normal ring. Next we show that the same method gives further examples of bad local domains, e.g., a two-dimensional normal local ring which is analytically ramified and a three-dimensional local domain whose derived normal ring is not noetherian. Since all claims of this section are easily verified, we omit their proofs. For detail, see [12, sections 5 and 6].

(4.1) (cf. [4, Proposition 1, Example 1]) Let k be a field of characteristic 2 . Letting X_i be indeterminates, we set $R_i = k[X_i^2 , X_i^3]$ with (fixed) maximal ideal $p_i = (X_i^2 , X_i^3)$ $(i = 1, 2, \ldots)$. Put $R' = \bigotimes_{i} {}_k R_i$ and $S = R' - \bigcup_i p_i R'$ (a multiplicatively closed set). Let $R = R'_S$. Then

(4.1.1) R is a one-dimensional noetherian domain with field of quotients $K = k(X_1, X_2, \dots)$, and

(4.1.2) for each maximal ideal q_i of R , $R_{q_i} = K_i[X_i^2, X_i^3]_{(X_i^2, X_i^3)}$ with certain extension field K_i of k.

Hence, by definition

(4.1.3) R is a G-ring.

Let $\overline{R}$ be the derived normal ring of R . Then

(4.1.4) $\overline{R} = R[X_1, X_2, \dots]$, and

(4.1.5) the set $\overline{R}^2 = \{ r^2 \mid r \in \overline{R} \}$ is contained in R .

Moreover, for any maximal ideal q of R

(4.1.6) R_q is not normal, but the derived normal ring $\overline{R}_q$ of R_q is a finite R_q-module (cf. (4.1.3)).

(4.2) Letting T be another indeterminate, we set

(4.2.0) $A = R[[T]]$, $C = \overline{R}[[T]]$ and $\omega = \sum_{j=1}^{\infty} X_j T^j$.

Then

(4.2.1) $\omega \notin Q(A)$, but $\omega^2 \in A$.

Let m be a maximal ideal of A and $q = m \cap R$.

Then, as $\overline{R}_q$ is a finite R_q-module (cf. (4.1.6))

(4.2.2) $\omega \in Q((A_m)\hat{\ }) = Q((R_q)\hat{\ }[[T]])$.

Therefore

(4.2.3) the $TR[T]$-adic completion A of a G-ring $R[T]$ is not an N-ring.

(4.3) Remark. With notation as above, we see

(4.3.0) A is a two-dimensional noetherian domain,

(4.3.1) A is complete and separated in the TA-adic topology, and

(4.3.2) $A/TA = R$ is a G-ring.

Hence A/a is a G-ring, for any non-zero ideal a of A.

(4.4) With notation as in (4.2), let

(4.4.0) $B = A[\omega]$, M a maximal ideal of B, b a non-zero element of M and $\overline{B_M}$ the derived normal ring of B_M.

Then

(4.4.1) the bB_M-adic completion B_M^* of B_M is not reduced, and

(4.4.2) $\overline{B_M}$ is a regular local ring.

Hence, for each height-one prime ideal P of B_M

(4.4.3) B_P is analytically ramified.

(4.5) Example. With notation as above, let

(4.5.0) $D = \overline{B_M} \cap B_M[\frac{1}{b}]$ (= the integral closure of B_M in $B_M[\frac{1}{b}]$).

Then

(4.5.1) D is not noetherian.

This gives an example of a two-dimensional local domain which has (an infinite number of) non-noetherian (quasi-local) over-rings between the domain and its derived normal ring.

(4.6) We make a minor change of notation. Let R , K and $\overline{R}$ be the same as in (4.1). We use Y_i , Z_j for X_{2i-1} , X_{2j} (i,j = 1, 2, ...). Letting T , U be two indeterminates, we set

(4.6.0) $A = R[[T,U]]$, $C = \overline{R}[[T,U]]$, $\omega_1 = \sum_{i=1}^{\infty} Y_i T^i$, $\omega_2 = \sum_{j=1}^{\infty} Z_j U^j$ and $\omega = \omega_1 + \omega_2$.

Let $P = (T,U)A$. Then

(4.6.1) A_P is a two-dimensional regular local ring.

(4.7) Example. With notation as above, let

(4.7.0) $B = A[\omega]$ and Q the prime ideal of B such that $Q \cap A = P$, then

(4.7.1) B_Q is normal.

Hence, this gives an example of a two-dimensional normal local ring which is analytically ramified.

(4.8) Example. With notation as above, let M be a maximal ideal of B . Then

(4.8.1) B_M is a three-dimensional local domain whose derived normal ring $\overline{B_M}$ is not noetherian.

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