数理解析研究所講究録
第 401 巻 1980 年 48-65

THE EXACT DEGREE OF PRECISION OF GENERALIZED GAUSS KRONROD INTEGRATION RULES

Philip Rabinowitz
Department of Applied Mathematics
The Weizmann Institute of Science
Rehovot, Israel

1. Introduction

In this paper we shall consider the Kronrod extensions (KE) to the Gauss-Gegenbauer integration rules (GGIR) and the Lobatto-Gegenbauer rules (LGIR). The Gegenbauer polynomials, $C_n^\mu(x)$, $\mu > -\frac{1}{2}$, are those polynomials which are orthogonal with respect to the weight function $w(x;\mu) \equiv (1-x^2)^{\mu-\frac{1}{2}}$ and have the following normalization [4, p. 174]

(1) $$\int_{-1}^{1} w(x;\mu)C_n^\mu(x)C_m^\mu(x)dx = \delta_{nm}\, h_{n\mu}$$

where

(2) $$h_{n\mu} = \pi^{\frac{1}{2}}\Gamma(n+2\mu)\Gamma(\mu+\tfrac{1}{2})/(n+\mu)n!\Gamma(\mu)\Gamma(2\mu)$$

which implies that $C_n^\mu(x) = k_{n\mu}x^n + \ldots$ where

(3) $$k_{n\mu} = 2^n\,\Gamma(n+\mu)/n!\,\Gamma(\mu) \quad .$$

$C_n^\mu(x)$ is even (odd) if n is even (odd). Special cases of $C_n^\mu(x)$, perhaps with a different normalization, are $T_n(x)$, the Chebyshev polynomials of the first kind ($\mu = 0$) , $P_n(x)$, the Legendre polynomials ($\mu = 1/2$) , and $U_n(x)$, the Chebyshev polynomials of the second kind ($\mu = 1$) .

The n-point GGIR is given by

$$(4) \qquad If \equiv \int_{-1}^{1} w(x;\mu) f(x)\,dx = \sum_{i=1}^{n} w_i\, f(x_i) + c_{n\mu} M_{2n}(f)$$

where we have omitted the dependence of w_i and x_i on μ and n , x_i are the zeros of $C_n^{\mu}(x)$,

$$(5) \qquad c_{n\mu} = 2^{2n}\, h_{n\mu}/k_{n\mu}^2$$

and $M_j(f)$ is defined to be equal to $f^{(j)}(\xi)/2^j j!$ for some $\xi \in (-1,1)$. The corresponding LGIR has n+1 points and is given by

$$(6) \qquad If = \sum_{i=1}^{n+1} \bar{w}_i\, f(\bar{x}_i) + \bar{c}_{n\mu} M_{2n}(f)$$

where the $\bar{x}_i$ are the zeros of $(1-x^2)\, C_{n-1}^{\mu+1}(x)$ and

$$(7) \qquad \bar{c}_{n\mu} = -\frac{2^{2n}\, h_{n-1,\mu+1}}{k_{n-1,\mu+1}^2} = -4\, c_{n-1,\mu+1}$$

Since the weights of the integration rules considered do not play a part in the discussion, we shall not treat them here except to remark that Monegato [9, 10] has shown that the weights u_i in (8) below are positive for $0 \leq \mu \leq 1$ and the v_i , for $0 \leq \mu \leq 2$.

The KEGGIR is given by

$$(8) \qquad If = \sum_{i=1}^{n} u_i \; f(x_i) + \sum_{i=1}^{n+1} v_i \; f(y_i) + E_{p_n}(f)$$

where $E_s(f) = 0$ if f is a polynomial of degree < s and $p_n = 2[(3n+3)/2]$. The y_i are the zeros of a certain polynomial $E_{n+1,\mu}(x)$ which we shall study in the next section. For the moment we state a result of Szegö [16] that for $0 \leq \mu \leq 2$, the y_i are real, lie in [-1,1] and we separated by the x_i . (For $\mu \neq 0$, the y_i lie in (-1,1).) The corresponding KELGIR is given by

$$(9) \qquad If = \sum_{i=1}^{n+1} \bar{u}_i \; f(\bar{x}_i) + \sum_{i=1}^{n} \bar{v}_i \; f(\bar{y}_i) + E_{q_n}(f)$$

where $q_n = 2[(3n+2)/2]$ and the $\bar{y}_i$ are the zeros of $E_{n,\mu+1}(x)$. Thus, taking into account that $\mu > -\frac{1}{2}$, we see that practical KEGGIR's exist for $0 \leq \mu \leq 2$ and KELGIR's, for $-\frac{1}{2} < \mu \leq 1$.

The first one to discover a KEGGIR was Kronrod [7] who dealt with the case $\mu = 1/2$, the Gauss-Legendre or standard Gauss rule. Subsequently, Patterson [13] , Piessens and Branders [14] and Monegato [11] improved on Kronrod's original work and extended his results to the usual Lobatto case ($\mu = 1/2$) . Barrucand [2] was the first to point out the connection between the KE's and the Szegö polynomials $E_{n+1,\mu}(x)$. KE's to other integration rules are discussed by Baratella [1], Kahaner and Monegato [5], Monegato [9, 12] and Ramskii [15] .

In the entire literature on this subject, it is stated that the KE's have error terms which vanish for polynomials of degree less than p_n (Gauss) or q_n (Lobatto), and in Kronrod's tables, he gives the error in the integration of x^{p_n} by the KEGGIR with $\mu = 1/2$. However, nowhere is it proved that these KE's are of exact degree p_n-1 or q_n-1 as the case may be, that is, that there exists a polynomial of degree p_n or q_n for which the corresponding KE is not exact. Indeed, such a statement is not true for all μ. Thus, as Monegato [9] points out, the KE of the n-point GGIR with $\mu = 0$, the first Gauss-Chebyshev rule, is exact for polynomials of degree $\leq 4n-1$ and in fact is identical with the KE of the corresponding (n+1)-point LGIR, being the (2n+1)-point LGIR, the first Lobatto-Chebyshev rule. Furthermore, the KE of the n-point GGIR with $\mu = 1$, the second Gauss-Chebyshev rule, is exact for polynomials of degree $\leq 4n+1$ and in fact, is identical with the (4n+1)-point GGIR. In the present work, we shall show that, except for $\mu = 0,1$ in the GGIR case and $\mu = 0$ in the LGIR case, we have the result that the exact precision of the KEGGIR is p_n-1 while that of the KELGIR is q_n-1. Furthermore, if these rules are of simplex type, i.e. if we can express the error term in the form $K_{n\mu} f^{(p_n)}(\xi)$ or $K_{n\mu} f^{(q_n)}(\xi)$, which we have not been able to prove, then we have the following result

$$(10) \qquad If = \sum_{i=1}^{n} u_i\, f(x_i) + \sum_{i=1}^{n+1} v_i\, f(y_i) + d_{n\mu} c_{n\mu} M_{p_n}(f)$$

$$(11) \qquad If = \sum_{i=1}^{n+1} \bar{u}_i\, f(\bar{x}_i) + \sum_{i=1}^{n} \bar{v}_i\, f(\bar{y}_i) + d_{n-1,\mu+1} \bar{c}_{n\mu} M_{q_n}(f)$$

where $d_{n\mu}$ is easily computable and does not vanish for $0 < \mu \le 2$, $\mu \neq 1$, and all $n \geq 2$. For $\mu = 2$ we have the explicit expression

$$(12) \qquad d_{n\mu} = \begin{cases} -\dfrac{2}{n+3}\left(\dfrac{n+1}{n+3}\right)^m & n \text{ even} \\ -4(n+2)(n+1)^{m-1}/(n+3)^{m+1} & n \text{ odd} \end{cases}$$

where $m = [(n+1)/2]$.

2. The Szegö Polynomials $E_{n+1,\mu}$

We give here the main results of Szegö with some minor modification of his notation and refer to [16] for details. See also Davis and Rabinowitz [3, pp 82-89] and Monegato [11].

The Gegenbauer function of the second kind, $Q_n^\mu(z)$, defined by

$$(13) \qquad Q_n^\mu(z) = \frac{\Gamma(2\mu)}{2\Gamma(\mu+\frac{1}{2})} \int_{-1}^{1} w(t;\mu) \frac{C_n^\mu(t)}{z-t}\, dt$$

$$= \frac{\Gamma(2\mu)}{2\Gamma(\mu+\frac{1}{2})} z^{-n-1} \sum_{i=0}^{\infty} \beta_i z^{-2i}$$

where

$$(14) \qquad \beta_i = \int_{-1}^{1} w(t;\mu)\, C_n^\mu(t) t^{n+2i} dt \ , \quad i = 0,1,\ldots$$

is analytic in the entire complex plane with a slit on the closed interval $[-1,1]$. Hence

$$(15)\qquad \frac{1}{Q_n^\mu(z)} = z^{n+1} \sum_{i=0}^{\infty} \gamma_i \, z^{-2i} = E_{n+1,\mu}(z) + \delta_1 \, z^{-1} + \delta_2 z^{-2} + \ldots.$$

defining the polynomial $E_{n+1,\mu}(z)$ which is even (odd) for n odd (even). Thus,

$$(16)\qquad Q_n^\mu(z)\, E_{n+1,\mu}(z) = 1 + b_1 \, z^{-n-2} + b_2 \, z^{-n-3} + \ldots$$

and by the argument given in [16] or [3]

$$(17)\qquad Q_n^\mu(z)\, E_{n+1,\mu}(z) = 1 + \sum_{i=0}^{n} c_i \, Q_{n+1+i}^\mu(z)$$

for certain constants $c_o,\ldots,c_n$ depending on μ and n. Since $Q_n^\mu(z)$ is an odd (even) function if n is even (odd), we have that $Q_n^\mu(z)\, E_{n+1,\mu}(z)$ is always an odd function which implies that $c_o = 0$ if n is odd.

Now the functions of the second kind satisfy the following relations:

$$(18)\qquad \lim_{\varepsilon\to 0} (Q_n^\mu(x+i\varepsilon)-Q_n^\mu(x-i\varepsilon)) = -i\pi \frac{\Gamma(2\mu)}{\Gamma(\mu+\frac{1}{2})} w(x;\mu)\; C_n^\mu(x)$$

$$(19)\qquad \lim_{\varepsilon\to 0} (Q_n^\mu(x+i\varepsilon)+Q_n^\mu(x-i\varepsilon)) = 2\, \tilde{Q}_n^\mu(x)$$

where $\tilde{Q}_n^\mu(x)$ is defined on the segment $[-1,1]$. Hence

$$(20)\qquad C_n^\mu(x)\, E_{n+1,\mu}(x) = \sum_{i=0}^{n} c_i C_{n+1+i}^\mu(x)$$

and

(21) $$\tilde{Q}_n^{\mu}(x)E_{n+1,\mu}(x) = 1 + \sum_{i=0}^{n} c_i \tilde{Q}_{n+1+i}^{\mu}(x)$$

From (20) it follows that

(22) $$\int_{-1}^{1} w(x;\mu)\ C_n^{\mu}(x)\ E_{n+1,\mu}(x)\ x^k dx = 0 \quad , \quad k = 0,1,2,\dots,n$$

so that by the theorem in [3, p. 77], an interpolatory integration rule based on the zeros of $C_n^{\mu}(x)$ and $E_{n+1,\mu}(x)$ is exact for all polynomials of degree $\leq 3n+1$ which forms the basis for KEGGIR's.

Now, it can be shown that

(23) $$Q_n^{\mu}(z) = \gamma_{n\mu} w^{-n-1} F(1-\mu, n+1; n+\mu+1; w^{-2})$$

$$= \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu}\ w^{-n-1-2j}$$

where $z = \frac{1}{2}(w+w^{-1})$, $\gamma_{n\mu} = \sqrt{\pi}\ \Gamma(n+2\mu)/\Gamma(n+\mu+1)$, $F(a,b;c;z)$ is the usual hypergeometric function, $f_{o\mu} = 1$,

(24) $$f_{j\mu} = (1-\mu/j)(1-\mu/n+\mu+j)f_{j-1,\mu} \quad ,$$

and we have not shown the dependence on n of the $f_{j\mu}$.

Setting $w = e^{-i\theta}$ and $x = \cos\theta$, we get that

(25) $$\tilde{Q}_n^{\mu}(x) = \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} T_{n+1+2j}(x) \quad .$$

Since $E_{n+1,\mu}(x)$ contains only even or odd powers of x, we can write $E_{n+1,\mu}(x)$ in the form

$$(26) \qquad E_{n+1,\mu}(x) = \sum_{i=0}^{m-1} \lambda_{i\mu} T_{n+1-2i}(x) + \begin{cases} \lambda_{m\mu} T_1(x), & n \text{ even} \\ \frac{1}{2}\lambda_{m\mu}, & n \text{ odd} \end{cases}$$

To determine the coefficients $\lambda_{i\mu}$, we equate in view of (21) and (25) the coefficients of $T_k(x)$, $k = 1,\dots,n+1$ in the product

$$(27) \qquad \tilde{Q}_n^{\mu}(x)\, E_{n+1,\mu}(x) = \gamma_{n\mu}\Big(\sum_{j=0}^{\infty} f_{j\mu} T_{n+1+2j}(x)\Big)\Big(\sum_{i=0}^{m}{}' \lambda_{i\mu} T_{n+1-2i}(x)\Big)$$

to zero and the coefficient of $T_o(x)$ to unity. Here the prime means that if n is odd, we replace $\lambda_{m\mu}$ by $\frac{1}{2}\lambda_{m\mu}$. Since $T_r(x)T_s(x) = \frac{1}{2}(T_{r+s}(x) + T_{|r-s|}(x))$, we see that the $\lambda_{i\mu}$ must satisfy the following equations

$$(28) \qquad \begin{aligned} &\lambda_{o\mu} = 2\gamma_{n\mu}^{-1} \\ &\sum_{i=0}^{k} f_{i\mu}\lambda_{k-i,\mu} = 0 \qquad\qquad k = 1,\dots,m \end{aligned}$$

Following Monegato [11], we define $\alpha_{i\mu} = \lambda_{i\mu}/\lambda_{o\mu}$, so that $\alpha_{o\mu} = 1$, $\alpha_{1\mu} = -f_{1\mu}$ and

$$(29) \qquad \alpha_{k\mu} = -f_{k\mu} - \sum_{i=1}^{k-1} f_{i\mu}\alpha_{k-i,\mu} \qquad\qquad k = 2,\dots,m$$

From this, we see that the $\alpha_{i\mu}$ are the first $m+1$ coefficients in the series

$$(30) \qquad \sum_{i=0}^{\infty} \alpha_{i\mu} u^i = \{\sum_{j=0}^{\infty} f_{j\mu} u^j\}^{-1}$$

so that we can also use (29) for indices $k > m$. Here also we have not indicated the dependence on n of the $\lambda_{i\mu}$ and $\alpha_{i\mu}$.

3. The Exact Degree of Precision of KEGGIR's and KELGIR's

Let us define

$$(31) \qquad f_k(x) = C_n^{\mu}(x)\, E_{n+1,\mu}(x)\, C_{n+1+k}^{\mu}(x) , \qquad k = 0,\dots,n .$$

Then from (20) it follows that $If_k = c_k h_{n+1+k,\mu}$. Since the KEGGIR applied to $f_k(x)$ vanishes, we have from (8) that $E_{p_n}(f_k) = c_k h_{n+1+k,\mu}$ so that the exact precision of the KEGGIR is determined by the first index for which $c_k \neq 0$. We now show that for $0 < \mu \leq 2$, $\mu \neq 1$, $c_o \neq 0$ for n even and $c_1 \neq 0$ for n odd.

Consider first the case n even. Substituting (25) and (27) into (21) and equating the coefficients of $T_{n+2}(x)$, we find that

$$(32) \qquad c_o \gamma_{n+1,\mu} = \frac{\gamma_{n\mu}}{2} \{\lambda_{m\mu} f_{o\mu} + \lambda_{m\mu} f_{1\mu} + \lambda_{m-1,\mu} f_{2\mu} + \dots + \lambda_{o\mu} f_{m+1,\mu}\}$$

$$= \alpha_{m\mu} + \alpha_{m\mu} f_{1\mu} + \alpha_{m-1,\mu} f_{2\mu} + \dots + \alpha_{1\mu} f_{m\mu} + f_{m+1,\mu} = \alpha_{m\mu} - \alpha_{m+1,\mu} .$$

Thus, it suffices to show that $\alpha_{m\mu} - \alpha_{m+1,\mu}$ does not vanish. In fact, we shall show that the $\alpha_{i\mu}$ are strictly monotonic. For $0 < \mu < 1$, the sequence $\{f_{j\mu}\}$ is completely monotonic, i.e. $(-1)^k \Delta^k f_{j\mu} > 0$ for all j and k [17, p. 137]. Hence, by a theorem of Kaluza [6], the sequence $\{-\alpha_{i+1,\mu}\}$ is also completely monotonic and hence strictly monotonic. For $1 < \mu < 2$, the sequence $\{-f_{j+1,\mu}\}$ is completely monotonic. From this it follows by some results in [6] that

$$\frac{\alpha_{i-1,\mu}}{\alpha_{i\mu}} > \frac{\alpha_{i\mu}}{\alpha_{i+1,\mu}} \quad , \quad i = 1,2,\dots \quad .$$

Since $\sum_{i=0}^{\infty} \alpha_{i\mu}$ converges, and in fact equals $\{F(1-\mu,n+1;n+\mu+1;1)\}^{-1}$, it follows that the sequence $\{\alpha_{i\mu}\}$ is strictly monotonic. For $\mu = 2$, Szegö [16] gives an explicit expression for the $\lambda_{i\mu}$,

$$(33) \qquad \lambda_{i2} = \frac{2}{\sqrt{\pi}} \; \frac{1}{n+3} \left(\frac{n+1}{n+3}\right)^i \quad , \qquad i = 0,1,\dots$$

which again shows that the α_{i2} are strictly monotonic.

We now consider the case n odd. Proceeding as before, this time equating the coefficients of $T_{n+3}(x)$, we find that

$$(34) \quad c_1 \gamma_{n+2,\mu} = \frac{\gamma_{n\mu}}{2} \{\lambda_{m\mu} f_{1\mu} + \lambda_{m-1,\mu} f_{o\mu} + \lambda_{m-1,\mu} f_{2\mu} + \lambda_{m-2,\mu} f_{3\mu} + \dots + \lambda_{o\mu} f_{m+1,\mu}$$

$$= \alpha_{m-1,\mu} + \alpha_{m\mu} f_{1\mu} + \alpha_{m-1,\mu} f_{2\mu} + \dots + \alpha_{1\mu} f_{m\mu} + f_{m+1,\mu} = \alpha_{m-1,\mu} - \alpha_{m+1,\mu}$$

Since the $\alpha_{i\mu}$ are strictly monotonic, it follows that $c_1 \neq 0$.

For $\mu = 0$, $f_{jo} = 1, j = 0,1,2,\dots$ so that $\lambda_{oo} = -\lambda_{10} = 2n/\pi^{\frac{1}{2}}$, $\lambda_{i0} = 0$, $i > 1$ and $E_{n+1,0} = \frac{2\pi}{\pi^{\frac{1}{2}}} \{T_{n+1}(x) - T_{n-1}(x)\}$, $n \geq 2$.

Hence

(35) $C_n^o(x)E_{n+1,0}(x) = k_1T_n\{T_{n+1}-T_{n-1}\} = \frac{k_1}{2}\{T_{2n+1}-T_{2n-1}\} = k_2(1-x^2)U_{2n-1} =$

$$= k_3(1-x^2)C_{2n-1}^1(x)$$

and the zeros of $C_n^o(x)\ E_{n+1,0}(x)$ are the abscissas of the (2n+1)-point LGIR for the weight $w(x;0)$ which is of exact precision $4n-1$, as can also be seen from the fact that c_{n-2} is the first c_k which does not vanish.

For $\mu = 1$, $f_{01} = 1$, $f_{j1} = 0$, $j > 0$ so that $\lambda_{01} = \frac{2}{\sqrt{\pi}}$, $\lambda_{i1} = 0$, $i > 0$ and $E_{n+1,1}(x) = \frac{2}{\sqrt{\pi}}\ T_{n+1}(x)$. Hence

(36) $C_n^1(x)\ E_{n+1,1}(x) = k_1'\ U_n(x)T_{n+1}(x) = k_2'\ C_{2n+1}^1(x)$

and the zeros of $C_n^1(x)\ E_{n+1,1}(x)$ are the abscissas of the (2n+1)-point GGIR for the weight $w(x;1)$ which is of exact precision $4n+1$ and which also follows from the fact that c_n is the first c_k which does not vanish.

In the case of the KELGIR, we define

(37) $\overline{f}_k(x) = (1-x^2)C_{n-1}^{\mu+1}(x)\ E_{n,\mu+1}(x)C_{n+k}^{\mu+1}(x)$, $k = 0,1,\ldots n$

so that $I\overline{f}_k = c_k\ h_{n+k,\mu+1}$. Hence, since $c_o = c_o(n-1,\mu+1) \neq 0$ for $n-1$ even, i.e. for n odd, while $c_1 \neq 0$ for $n-1$ odd, we have that the (2n+1)-point KELGIR is of exact precision $3n+1$ for n even and $3n$ for n odd, provided that $\mu \neq 0$. For $\mu = 0$, we have as before that $E_{n1}(x) = \frac{2}{\pi^{\frac{1}{2}}}\ T_n(x)$ so that

(38) $$(1-x^2)\ C^1_{n-1}(x)\ E_{n1}(x) = \hat{k}_1 (1-x^2)\ C^1_{2n-1}(x)$$

whose zeros are again the abscissas of the (2n+1)-point LGIR for the weight $w(x;0)$.

If we now define

(39) $$d_{n\mu} = \begin{cases} \alpha_{m\mu}-\alpha_{m+1,\mu} & n \text{ even} \\ \alpha_{m-1,\mu}-\alpha_{m+1,\mu} & n \text{ odd}, \ m = [(n+1)/2] \end{cases}$$

we have that for the Gauss case

(40) $$d_{n\mu} = \begin{cases} c_o\ \gamma_{n+1,\mu} & n \text{ even} \\ c_1\ \gamma_{n+2,\mu} & n \text{ odd} \end{cases}$$

while for the Lobatto case

$$d_{n-1,\mu+1} = \begin{cases} c_o\ \gamma_{n,\mu+1} & n \text{ even} \\ c_1\ \gamma_{n+1,\mu+1} & n \text{ odd} \end{cases}$$

where we have suppressed the dependence of c_o and c_1 on n and μ. This lead us immediately to formulas (10) and (11). For example, applying (8) with n even to $f_o(x)$, we have that

(41) $$c_o\ h_{n+1,\mu} = K_{n\mu} k_{n\mu} 2\gamma^{-1}_{n\mu} 2^n k_{n+1,\mu} (3n+2)!$$

so that

(42) $$K_{n\mu} = \frac{d_{n\mu}}{\gamma_{n+1,\mu}}\ \frac{h_{n+1,\mu}\gamma_{n\mu}}{2^{n+1} k_{n\mu} k_{n+1,\mu} (3n+2)!} = \frac{d_{n\mu}\ c_{n\mu}}{2^{p_n}\ p_n!} \ .$$

For n odd, we consider $f_1(x)$ while in the Lobatto case we work with $\overline{f}_o(x)$ and $\overline{f}_1(x)$.

4. Remarks

a. Monegato [11] gives an error bound for KEGGIR's with $0 < \mu < 1$. We shall show how to improve this bound slightly and extend it to the case $1 < \mu < 2$ as well as to KELGIR's with $-\frac{1}{2} < \mu \leq 1$, $\mu \neq 0$.

For n even, Monegato writes the error $E_{p_n}(f)$ for $f \in C^{3n+2}[-1,1]$ in the form

$$(43) \qquad E_{p_n}(f) = \frac{2^{-2n}}{k_{n\mu}(3n+2)!} \int_{-1}^{1} w(x;\mu)\, C_n^{\mu}(x)\, (\overline{E}_{n+1,\mu}(x))^2\, f^{(3n+2)}(\xi_x)\, dx$$

where

$$(44) \qquad \overline{E}_{n+1,\mu}(x) = E_{n+1,\mu}(x)/\lambda_{o\mu} = \sum_{i=0}^{m}{}' \alpha_{i\mu}\, T_{n+1-2i}(x) .$$

Hence

$$(45) \qquad |E_{p_n}(f)| \leq \frac{\pi\, \Gamma(n+2\mu)\, B^2_{n+1,\mu}}{2^{3n+2\mu-1}\, p_n!\, \Gamma(\mu+1)\Gamma(n+\mu)}\, M_{p_n}$$

where $M_s = \max_{-1\leq x\leq 1} |f^{(s)}(x)|$ and $B_{n+1,\mu} = \max_{-1\leq x\leq 1} |\overline{E}_{n+1,\mu}(x)|$.

For $0 < \mu < 1$, Monegato states that $B_{n+1,\mu} < 2$ and replaces $B_{n+1,\mu}$ by 2 in (45). Now while this bound is the best available for $0 < \mu \leq \frac{1}{2}$, we can improve on it for $\frac{1}{2} < \mu < 1$. In addition, a bound on $B_{n+1,\mu}$ is also available for $1 < \mu \leq 2$. This follows from our observation above that

$$(46) \quad \sum_{i=0}^{\infty} \alpha_{i\mu} = \{F(1-\mu,n+1;n+\mu+1,1)\}^{-1} \equiv T_{n\mu} = \frac{\Gamma(\mu)\Gamma(n+2\mu)}{\Gamma(n+\mu+1)\Gamma(2\mu-1)} , \quad \mu > \frac{1}{2}, \ \mu \neq 1,2 .$$

Now for $\frac{1}{2} < \mu < 1$, $\alpha_{o\mu} = 1$, $\alpha_{i\mu} < 0$, $i > 0$. Since

$$B_{n+1,\mu} \leq \sum_{i=0}^{m} |\alpha_{i\mu}| = 1 - \sum_{i=1}^{m} \alpha_{i\mu} < 1 - \sum_{i=1}^{\infty} \alpha_{i\mu}$$

it follows that $B_{n+1,\mu} < 2-T_{n\mu} < 2$. For $1 < \mu < 2$, we have that $\alpha_{i\mu} > 0$, all i . Hence $B_{n+1,\mu} \leq \sum_{i=0}^{m} \alpha_{i\mu} < T_{n\mu}$. For $\mu = 2$, $\sum_{i=0}^{\infty} \alpha_{i2} = \left(1 - \frac{n+1}{n+3}\right)^{-1} = \frac{n+3}{2} > B_{n+1,2}$.

For n odd, using classical arguments, we have the same bound.

In the Lobatto case, we have similarly for n odd that

$$(47) \quad E_{q_n}(x) = \frac{2^{2-2n}}{k_{n-1,\mu+1}(3n)!} \int_{-1}^{1} w(x;\mu+1)C_{n-1}^{\mu+1}(x) \, (\overline{E}_{n,\mu+1}(x))^2 f^{(3n)}(\overline{\xi}_x)dx$$

whence

$$(48) \quad |E_{q_n}(f)| \leq \frac{\pi \, \Gamma(n+2\mu+1) \, B_{n,\mu+1}^2}{2^{3n+2\mu-2} \, q_n! \, \Gamma(n+\mu)\Gamma(\mu+2)} M_{q_n}$$

where for $-\frac{1}{2} < \mu < 0$, $B_{n,\mu+1} < 2-T_{n-1,\mu+1}$ and for $0 < \mu < 1$,

$B_{n,\mu+1} < T_{n-1,\mu+1}$. For $\mu = 1$, $B_{n2} < \frac{n+2}{2}$. As before, the same bound holds for n even.

b. The Fourier-Gegenbauer coefficients of a function $f(x)$ are defined by

(49) $$FG_{n\mu}(f) = h_{n\mu}^{-1} \int_{-1}^{1} w(x;\mu)\, C_n^{\mu}(x) f(x)\,dx\ , \quad n = 0,1\ldots\ .$$

As Barracund [2] points out, the integral is most efficiently evaluated by a (2n+1)-point KEGGIR applied to the function $C_n^{\mu}(x)f(x)$ which reduces to the $(n+1)$-point formula

(50) $$FG_{n\mu}(f) \simeq h_{n\mu}^{-1} \sum_{i=1}^{n+1} v_i\, C_n^{\mu}(y_i) = \sum_{i=1}^{n+1} \tilde{v}_i\, f(y_i)\ .$$

For $\mu \neq 0,1$, we get a rule which is exact for polynomials of degree $< p_n - n$, which is the best possible. For assume that there existed an (n+1)-point rule, say

(51) $$FG_n(f) \simeq \sum_{i=1}^{n+1} \hat{v}_i\, f(\hat{y}_i)$$

exact for polynomials of degree $p_n - n$, n even. This would imply that

(52) $$\int_{-1}^{1} w(x;\mu) C_n^{\mu}(x) E_{n+1,\mu}(x) \prod_{i=1}^{n+1} (x-\hat{y}_i)\,dx = 0$$

which contradicts our results above. Similarly for n odd.

For $\mu = 0$, the rule (50) is exact for polynomials of degree $\leq 3n-1$, a result which has already been reported in [8] . For $\mu = 1$, (50) is exact for polynomials of degree $\leq 3n+1$ which is the best possible result, so that the highest precision is achieved for Fourier-Chebyshev coefficients of the second kind. However, we should warn the user that the weights $\tilde{v}_i$ in (50) alternate in sign inasmuch as the v_i are positive and the zeros of $C_n^{\mu}(x)$ separate those of $E_{n+1,\mu}(x)$ so that the $C_n^{\mu}(y_i)$ alternate in sign.

REFERENCES

1. P. Baratella, Un'estensione ottimale della formula di quadratura di Radau, Rend. Sem. Mat. Univers. Politecn. Torino, v. 37(1979) pp. 147-158.

2. P. Barrucand, Integration numérique, abscisse de Kronrod-Patterson et polynômes de Szegö, C. R. Acad. Sc. Paris, Série A, v. 270(1970) pp. 336-338.

3. P. J. Davis and P. Rabinowitz, Methods of Numerical Integration, Academic Press, New York, 1975.

4. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, Vol. II, McGraw-Hill, New York, 1953.

5. D. K. Kahaner and G. Monegato, Nonexistence of extented Gauss-Laguerre and Gauss-Hermite quadrature rules with positive weights, ZAMP v. 29(1978) pp. 983-986.

6. T. Kaluza, Über die Koeffizienten reziproker Potenzreihen, Math. Zeit. v. 28(1928) pp. 161-170.

7. A. S. Kronrod, Nodes and Weights for Quadrature Formulae. Sixteen Place Tables. 'Nauka' Moscow, 1964; English Trans., Consultants Bureau, New York, 1965.

8. C. A. Micchelli and T. J. Rivlin, Turán formulae,and highest precision quadrature rules for Chebyshev coefficients,IBM J. Res. Develop. v. 16(1972) pp. 372-379.

9. G. Monegato, A note on extended Gaussian quadrature rules, Math. Comp. v. 30(1976) pp. 812-817.

10. G. Monegato, Positivity of the weights of extended Gauss-Legendre quadrature rules, Math. Comp. v. 32(1978) pp. 243-245.

11. G. Monegato, Some remarks on the construction of extended Gaussian quadrature rules, Math. Comp. v. 32(1978) pp. 247-252.

12. G. Monegato, An overview of results and questions related to Kronrod schemes, in Numerische Integration (G. Hämmerlin, ed.), ISNM 45, Birkäuser Verlag, Basel, 1979, pp. 231-240.

13. T. N. L. Patterson, The optimum addition of points to quadrature formulae, Math. Comp. v. 22(1967) pp. 847-856.

14. R. Piessens and M. Branders, A note on the optimal addition of absċissas to quadrature formulas of Gauss and Lobatto type, Math. Comp. v. 28 (1974) pp. 135-139.

15. Ju. S. Ramskii, The improvement of a certain quadrature formula of Gauss type, Vycisl. Prikl. Mat. (Kiev), v. 22(1974) pp. 143-146 (Russian).

16. G. Szegö, Über gewisse orthogonale Polynome, die zu einer oszillierenden Belegungsfunktion gehören, Math. Ann. v. 110(1934) pp. 501-513.

17. G. Szegö, Orthogonal Polynomials, Rev. Ed., American Mathematical Society, New York, 1959.