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ON A CLASS OF SINGULAR PSEUDO-DIFFERENTIAL OPERATORS

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In the past one or two decades there appeared extensive works on singular partial differential operators, see e.g. S. Alinhac [1],[2] and H. Tahara [13], [14], [15]. Among them the so-called Fuchsian type equations are most remarkable, see Baouendi and Goulaouic [4] and the author's paper [5]. But in the classical work of J. Hadamard on the construction of the fundamental solution, there appears non-Fuchsian partial differential equation whose properties are quite different from that of the Fuchsian type equations, see [8], [5], [6]. Hence it is desirable to study in more detail a class of singular pseudo-differential operators of the form

$$D_t u + A(x,t,D_x)u + t^{-m} B(x,D_x)u = f, \quad m\in\mathbb{N} \qquad (1)$$

where A, B are proper pseudo-differential operators of first order on a C^∞ manifold M (countable at infinity, t a parameter) with complete symbols $a(x,t,\xi)$ and $b(x,\xi)$. $b(x,\xi)$ is positively homogeneous in ξ of degree 1.

Our main result is, under certain conditions, equation (1) may be reduced to

$$D_t V + A(x,t,D_x)\, V = F. \qquad (2)$$

Now, let's explain our chief idea, but for the moment only formally.

Consider the equation

$$t^m D_t u + B(x,D_x)u = v \qquad m \text{ even} \tag{3}$$

and the diffeomorphism $R_t \setminus 0 \to R_\tau \setminus 0$

$$\tau = t^{1-m} / (1-m) , \tag{4}$$

or the equation pair

$$\begin{cases} t^m D_t u + B(x,D_x)u = v & t>0 \\ -t^m D_t u + B(x,D_x)u = v & t<0 \end{cases} \qquad m \text{ odd. } m>1 \tag{5}$$

and the diffeomorphism $R_t \setminus 0 \to R_\tau \setminus 0$

$$\begin{cases} \tau = t^{1-m} /(1-m) & t>0 \\ \tau = -(-t)^{1-m} / (1-m) & t<0 , \end{cases} \tag{6}$$

or when m=1 consider separately in $R_t^+ : t>0$ and $R_t^- : t<0$ the equation

$$tD_t u + B(x,D_x)u = v \tag{7}$$

and the diffeomorphism $R_t^{\pm} \setminus 0 \to R_\tau \setminus 0$

$$\begin{cases} \tau = \ell n t & t>0 \\ \tau = \ell n(-t) & t<0, \end{cases} \tag{8}$$

we shall have

$$D_\tau u + B(x,D_x)u = v \ , \qquad (9)$$

hence formally

$$u = \exp\ (-i\tau B)v \ . \qquad (10)$$

Substituting it into (1) gives

$$\frac{\partial v}{\partial \tau} + i\exp(i\tau B)\cdot A\cdot\exp(-i\tau B)v = \exp(i\tau B)f.$$

when A and B commute, hence A and $\exp(i\tau B)$ also commute, we have

$$\frac{\partial v}{\partial \tau} + iAv = F \ . \qquad (11)$$

(11) is an equation without singularity in the left hand side.

It is readily seen that the arguments above may be made rigorous once $\{\exp(-i\tau B)\}$ $(\forall\ \tau\in\mathbb{R})$ is defined rigorously and proved to be a group. This can be done when the manifold M is compact without boundary and $B(x,D_x)$ formally self-adjoint and elliptic as is well known in the spectral theory of self-adjoint operators. But there is another approach to this aim, i.e. to solve the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + iBu = 0 \\ u(0,x) = u_0(x). \end{cases} \qquad (12)$$

In fact, by "Fourier method", it is easy to see, the solution of Cauchy problem (12) is just $u = \exp(-i\tau B)u_0$.

But, the Cauchy problem is solvable not only when B is formally self-adjoint and elliptic. Hence, if we can find the fundamental solution or at least the parametrix $U(\tau)$ of the Cauchy probelm (12), its solution would just be

$$u(\tau,x) = U(\tau)u_0(x)$$

and we may use $U(\tau)$ as the exponential $\exp(-i\tau B)$.

In defining the parametrix of the Cauchy problem (12), behavior of the orbits of the Hamiltonian field

$$H_B = (b_\xi(x,\zeta)\partial_x, -b_x(x,\zeta)\partial_\xi)$$

is very important. In our case, we should at least assume its existance for all $\tau \in \mathbb{R}_\tau$ since τ near 0 corresponds to τ near infinity.

Now, we proceed to construct the parametrix for the Cauchy problem (12) first for small τ. Such contruction, as usual, amounts to seek an operator $U(\tau)$: $C_0^\infty(M) \to C^\infty(M)$ such that

$$\begin{cases} (D_\tau + B)\, U(\tau) \in S^{-\infty} \\ U(0) - 1 \in S^{-\infty} \end{cases} \tag{13}$$

Linearity of the problem allows us to use a partition of unity and reduce our problem to the case $M = \mathbb{R}^n$, i.e., to consider (13) in $\mathbb{R}_\tau \times \mathbb{R}_x^n$.

Now assume $U(\tau)$ to be a Fourier integral operator (F.I.O.)

$$U(\tau)f(x) = (2\pi)^{-n} \int \exp[iS(\tau,x,y,\eta)]q(\tau,x,\eta)f(y)dyd\eta \qquad f(x) \in C_0^\infty(\mathbb{R}^n) \tag{14}$$

with a distribution kernel

$$U(\tau,x,y) = (2\pi)^{-n}\int \exp\,[iS(\tau,x,y,\eta)]q(\tau,x,\eta)d\eta.$$

Let $S(\tau,x,y,\eta) = \Phi(\tau,x,\eta) - y\cdot\eta$, we have the following Cauchy problem for $\Phi(\tau,x,\eta)$

$$\left\{\begin{aligned} &\frac{\partial\Phi}{\partial\tau} + B(x,\Phi_x) = 0 \\ &\Phi(0,x,\eta) = x\cdot\eta\ . \end{aligned}\right. \tag{15}$$

In order to solve it, we turn to the Hamiltonian system

$$\frac{dx}{ds} = b_\eta(x,\eta), \quad \frac{d\eta}{ds} = -b_x(x,\eta). \tag{16}$$

Assuming the initial conditions

$$x|_{\tau=0} = x_0(z_1,\ldots,z_n)\ , \quad \eta|_{\tau=0} = \eta_0(z_1,\ldots,z_n)$$

where $\eta_1,\ldots,\eta_n$ are n parameters such that

$$\det\,(\tfrac{\partial x}{\partial z})_{\tau=0} \neq 0\ ,$$

we have solution for (16) as

$$x = x(\tau,z), \quad \eta = \eta(\tau,z) \quad z = (z_1,\ldots,z_n) \tag{17}$$

From classical theory of partial differential equations of first order, we have

$$\begin{aligned}\Phi(\tau,x) = (x_0\cdot\eta_0)(z) + \int_0^\tau [&\langle\eta(\rho,z), \frac{dx(\rho,z)}{d\rho}\rangle \\ &-B(x(\rho,z),\eta(\rho,z))]d\rho\ ,\end{aligned} \tag{18}$$

which is valid as far as $\det \frac{\partial(\tau,x)}{\partial(\tau,z)} = \det\,(\frac{\partial x}{\partial z}) \neq 0$, i.e., for $|\tau|$ sufficiently small.

The amplitude $q(\tau,x,\eta)$ is to be found in the class

$$q(\tau,x,\eta) \sim \sum_{k=0}^{\infty} q_k(\tau,x,\eta), \tag{19}$$

where $q_k(\tau,x,\eta)$ are positively homogeneous in η of degree $-k$. For q_k we have the so-called transport equations

$$\partial_\tau q_0 + \sum_{|\alpha|=1} b^{(\alpha)}(x,\Phi_x)\,\partial_x^\alpha q_0 + \sum_{|\alpha|=2} \frac{1}{\alpha!} b^{(\alpha)}(x,\Phi_x)\partial_x^\alpha \Phi \cdot q_0 = 0,$$

$$\partial_\tau q_k + \sum_{|\alpha|=1} b^{(\alpha)}(x,\Phi_x)\partial_x q_k + \sum_{|\alpha|=2} \frac{1}{\alpha!} b^{(\alpha)}(x,\Phi_x)\partial_x^\alpha \Phi \cdot q_k \tag{20}$$

$$+ R_k(q_0,\dots,q_{k-1}) = 0 \ , \qquad k\geq 1 \ ,$$

$$q_k(0,x,\eta) = \delta_{0k} \ . \tag{21}$$

Hence

$$q_0(\tau,x,\eta) = \sqrt{\left| \frac{J(0,z)}{J(\zeta,z)}\right|} \ \exp\left[\int_0^\tau \tfrac{1}{2}\mathrm{tr}\left(\frac{\partial^2 b(x,\Phi_x)}{\partial x \partial \eta}\right)d\tau\right] \tag{22}$$

Where $J(\tau,z) = \det \frac{\partial(\tau,x)}{\partial(\tau,z)}$. Similar results hold for q_k when $k\geq 1$. Thus (22) holds only when $\det \frac{\partial(\tau,x)}{\partial(\tau,z)} = \det\left(\frac{\partial x}{\partial z}\right) \neq 0$, i.e., when $|\tau|$ sufficiently small.

We shall now follow Maslov's line to contruct a parametrix valid for all τ. The intrinsic object connected with the Cauchy problem (15) is a Lagrangean manifold Λ^{n+1} of demension $n+1$ constructed in the following way. Through every point of an n-dimensional isotropic manifold $\Lambda^n \subset T^k(\mathbb{R}_\tau \times M)\setminus 0 = (\mathbb{R}_\tau \times \mathbb{R}_x^n) \times [(\mathbb{R}_E \times \mathbb{R}_p^n)\setminus 0]$ passes an orbit of the Hamiltonian vector field $H_B(x,p)$

$$\frac{d\tau}{ds} = 1, \quad \frac{dE}{ds} = 0$$

$$\frac{dx}{ds} = b_p(x,p), \quad \frac{dp}{ds} = -b_x(x,p)$$

these orbits then form Λ^{n+1}. Here $\Lambda^n \subset \{(\tau,x,E,p): E+b(x,p)=0\}$, E is the dual variable of τ. Later τ and E will be denoted by x_0 and p_0 respectively.

Maslov and Arnold [11], [13] proved that there exists a canonical atlas for Λ^{n+1}, such that the local coordinate in any chart (simply connected) should be of the form (x_I,p_J), where I and J are subsets of $\{0,1,\ldots,n\}$. $I \cup J = \{0,1,\ldots,n\}$, $I \cap J = \phi$. Those charts which are diffemorphic to $\mathbb{R}_x$ are called regular. Those whrerin is a point with no neighborhood diffeomorphic to $\mathbb{R}_x$ are called singular charts and such points singular points. The set of singular points $\Sigma(\Lambda^{n+1})$ is a cycle — singular cycle, which can always be assumed to be an n-dimentional submanifold of Λ^{n+1}.

In Maslov's work, x and p are put on a completely equal footing, hence we should modify the class of symbols of pseudo-differential operators in accordance.

Definition *Let $a(x,p)$ $C^\infty(\mathbb{R}_x \times \mathbb{R}_p)$ satisfy the following condition :*

There exists costant $C_{\alpha\beta}>0$ such that for all $(x,p) \in \mathbb{R}_x \times \mathbb{R}_p$

$$|\partial_x^\alpha \partial_p^\beta a(x,p)| \le C_{\alpha\beta} (1+|x|)^{m-|\alpha|}(1+|p|)^{m-|\beta|} \tag{23}$$

we say the symbol $a(x,p) \in M^m$.

Now let there exists a C^∞ positive 1-density $d\sigma$ on M, which is invariant under the action of H_B. Our main result is

Theorem 1 *If M is a C^∞ (countable at infity) manifold of dmension n with a C^∞ positive* 1*-density $d\sigma$ defined on it. Let $B(x,D_x)$ be a proper pseudo-differential operator of order* 1 *with a real symbol $b(x,p)\in M^1$ which is positively homogeneous in (x_I, p_J) of degree* 1 *for arbitrary complementary subsets I and J of $\{1,2,\ldots,n\}$ with $I\cap J = \phi$. Assume $H_B = (b_p(x,p)\partial_x$, $-b_x(x,p)\partial_p)$ defines a* 1*-parameter group of diddeomorphism and $d\sigma$ is invariant under it. Denote by Λ^{n+1} the Lagrangean manifold defined by the equation (15) with canonical coordinates (x,p). Suppose the following conditions of guantization hold*

(1) $\int_\gamma p dx = 0$ *for every closed path γ on Λ^{n+1};* (24)

(2) *The Maslov index on every cycle is o.* (25)

Under these conditions, a parametrix for arbitrary τ for the Cauchy problem (12) *exists.*

Outline of Proof. Let Ω_0 be a regular chart where

$$d\sigma = |dx|.$$

The phase function $\Phi(\tau, x, \eta)$ in (14) where $S_{(\tau,x,y,\eta)} = \Phi_{(\tau,x,\eta)} - y\cdot\eta$ is just the generating function of the Lagrangean Manifold Λ^{n+1} in this coordinate patch, which may be expressed as

$$\Phi_{\Omega_0}(\tau,x,\eta) = \int_{P_0}^{P} pdx \qquad (26)$$

where P_0 is a fixed point in Ω_0 and $P = (\tau, x, \eta)$. In Maslov's work [11], a pre-canonical operator on Ω_0

$$K_{\Omega_0}[q_{\Omega_0}(\tau,x,\eta)] = q_{\Omega_0}(\tau,x,\eta) \exp [i \int_{P_0}^{P} pdx]\sqrt{\left|\frac{d\sigma}{dx}\right|} \quad (27)$$

is defined. Here $q_{\Omega_0}(\tau,x,\eta)$ is just $q(\tau,x,\eta)$ constructed above. Thus the local parametrix (14) may be written as

$$[U_{\Omega_0}(\tau)f](Q) = (2\pi)^{-n} \int K_{\Omega_0}[q_{\Omega_0}(\tau,Q,\eta)] \hat{f}(\eta)d\eta, \quad (28)$$

$$Q = Q(x) .$$

If we follow the orbit of H_B and enter another coordinate patch Ω_1 of Λ^{n+1} with focal coordinate (x_I, p_J), denote the generating function of Λ^{n+1} on Ω_1 by Φ_{Ω_1}

$$\Phi_{\Omega_1}(\tau, Q, \eta) = \int_{Q_0}^{Q} pdx - \langle x_J(x_I, p_J), p_J \rangle ,$$

$$Q_0 \in \Omega_0 \cap \Omega_1$$

The pre-canonical operator K_{Ω_1} is

$$K_{\Omega_1}[q(\tau, Q, \eta)] = \exp \left(\frac{\pi i}{2}\alpha_1\right) q_{\Omega_1}(\tau, Q, \eta) \cdot$$

$$\exp [i\Phi_{\Omega_1}(\tau, Q, \eta)]\sqrt{\left|\frac{d\sigma}{dx_I dp_J}\right|}$$

$$Q = Q(x_I, p_J)$$

α_1 = Maslov's index of Ω_1

and we try to find a local parametrix $U_{\Omega_1}(\tau)$ of the Cauchy problem (13) as

$$[U_{\Omega_1}(\tau)f]\ (Q) = (2\pi)^{-n}\ F^{-1}_{p_J \to x_J} \int K_{\Omega_1}[q_{\Omega_1}(\tau,Q,\eta)]\hat{f}(\eta)d\eta, \quad (29)$$

$$Q = Q(x_I,\ p_J)\ ,$$

where $F^{-1}_{p_J \to x_J}$ is the inverse partial Fourier transform.

When (29) is substituted into equation (13), we shall have corresponding transport equation for $q_{\Omega_1}(\tau,\ Q,\ \eta)$, but the initial condition will no longer be (21), but with the values at $Q_0 \in \Omega_0 \cap \Omega_1$ of $q_{\Omega_0 k}(\tau,\ Q,\ \eta)$ as the initial values of $q_{\Omega_1 k}(\tau,\ Q,\ \eta)$.

It is known [11], at $Q = Q(x) = Q(x_I,\ p_J) \in \Omega_0 \cap \Omega_1$,

$$[U_{\Omega_0}(\tau)f](Q) = [U_{\Omega_1}(\tau)f](Q)\ . \quad (30)$$

Thus, we may piece together the local parametrices obtained on each coordinate patch as follows. Let $\{e_i\}$ be a partition of unity subordinated to the canonical atlas $\{\Omega_j\}$, define

$$q(\tau,Q,\eta) = \sum_j q_j(\tau,Q,\eta) = \sum_j e_j q_{\Omega_j}(\tau,Q,\eta),$$

$$K_\Lambda[q(\tau,Q,\eta)] = \sum_j F^{-1}_{p_{J_j} \to x_{J_j}} K_{\Omega_j}[e_j q_{\Omega_j}(\tau,Q,\eta)],$$

(Maslov's global cononical operator).
The parametrix for arbitrary τ will be

$$[U(\tau)f]Q = (2\pi)^{-n}\int K_{\Lambda}[q(\tau,Q,\eta)]\hat{f}(\eta)d\eta \ . \qquad (31)$$

In order (31) be well-defined, it is sufficient that

(i) $\int_{\gamma} pdx = 0$ on every cycle γ;

(ii) The Maslov's index for every cycle is 0.

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Now, it is easy to prove the following properties of $U(\tau)$.

Theorem 2 *If the Cauchy problem* (13) *has a unique (up to* mod $S^{-\infty}$*) solution* $U(\tau)$, *then*

$$U(\tau_1 + \tau_2) = U(\tau_1) \circ U(\tau_2), \quad (\text{mod } S^{-\infty}). \qquad (32)$$

Theorem 3 *If* $A(x, D_x)$ *is independent of* τ *and* $[A, B] = 0$, *then* $[A, U] = 0$. *Equalities are in a* mod $S^{-\infty}$ *sense.*

The proofs are omitted.

Now we turn to some concrete cases,

First let M be a compact C^{∞} manifold (countable at infinity) without boundary, $B(x,D_x)$ formally self-adjoint and elliptic. In this case $\{\exp(i\tau B)\}$ may be defined by spectral-theoretic method and is a group of unitary operators. It is easy to prove

Theorem 4 *If* A *and* B *commute,* A *and* $U(\tau)$ *also commute (both in a mod* $S^{-\infty}$ *sense)*

Proof is omitted.

Turn to the equation (1). Introduce the new unknown v by

$$u = \exp(-i\tau B)v(\tau,x) = U(\tau)v(\tau,x) \pmod{S^{-\infty}}, \qquad (33)$$

for $v(\tau,x)$ we have

$$D_t v + U^{-1}A(t,x,D_x)Uv = U^{-1}f \quad \pmod{S^{-\infty}}. \qquad (34)$$

Theorem 5 *If* A *and* B *commute, equation (1) may be reduced to*

$$D_t v + Av = U^{-1}f \qquad (35)$$

If $[A,B] \neq 0$, *but* $[[A,B],B] = 0$, *equation* (1) *may be reduced to*

$$D_t v + Av + \frac{i[A,B]}{(1-m)t^{m-1}} v = U^{-1}f \quad m>1, \qquad (36)$$

$$D_t v + Av + i\ln t\cdot[A,B]v = U^{-1}f \quad m=1,$$

With lower order singularity. All the equalities are taken in mod $S^{-\infty}$*sense.*

The proof is also straightforward.

In the second case, M is no longer assumed to be compact, but the symbol $b(x,\xi)$ of $B(x,D_x)$ is assumed to be real and the Hamiltonian of $b(x,\xi)$ is not the radial direction, i.e., H_B and $\xi\partial_\xi$ are not parallel. A typical example of this case is

$$t^m (D_t + A(t,x,D_x))u + D_{x_1}u = t^m f. \qquad (37)$$

We may use the method above or the partial Fourier transformation to construct its parametrix, but the most straightforward way is to introduce new variables, e.g., when $m>1$,

$$\tilde{x}_1 = x_1 - t^{1-m} / (1-m), \quad \tilde{t} = t, \quad \tilde{x}_j = x_j \quad j>1 \tag{38}$$

Then we have (still use the notations t and x)

$$\begin{aligned} &D_t u + A(t, x_1+t^{1-m}/(1-m), x_2, \ldots, x_n, D_x)u \\ &= F(t, x_1+t^{1-m}/(1-m), x_2, \ldots, x_n) \end{aligned} \tag{39}$$

In fact, in this case $B = D_{x_1} = \frac{1}{i}\partial_{x_1}$ is still a self-adjoint operator and its exponential is just the translation operator. What is more important is that we may use a canonical transformation such that microlocally equation (1) is equivalent to (37).

Theorem 6 *Under the conditions above, equation* (1) *may be reduced to an equation without singularity when* $[A,B] = 0$ (mod $S^{-\infty}$), *or reduced to an equation with lower singularity when* $[[A,B],B] = 0$ (mod $S^{-\infty}$).

Proof It is well known [7],[9] that there exists a unitary Fourier integral operator U such that microlocally

$$U^{-1}BU = D_{x_1} \quad (\text{mod } S^{-\infty})$$

Denote by $A_1(t,x,D_x) = U^{-1}AU$, we see

$$0 = U^{-1}[A,B]U = [A_1, D_{x_1}]$$

when $[A,B] = 0$. By computing the symbols , we have

$$D_{x_1} a_1(t,x,\xi) = 0, \quad (\text{mod } S^{-\infty}),$$

where $a_1(t,x,\xi)$ is the symbol of A_1, hence a_1 is independent of x_1. By introducing the new variables (38), we see equation (39) becomes

$$D_t u + A_1(t,x,D_x)u = f(t, x_1 + t^{1-m}/(1-m), x_2, \ldots, x_n). \tag{40}$$

When $m=1$, corresponding result will be obtained.

When $[[A,B], B] = 0$, it is easy to see

$$A(t,x,D_x) = A_1(t,x,D_x) + x_1 A_2(t,x,D_x) \ (\text{mod } S^{-\infty}),$$

where A_1 and A_2 have symbols independent of x_1 . Use the method above, we have

$$D_t u + A_1(t,x,D_x)u + [x_1 + t^{1-m}/(1-m)] A_2(x,t,D_x)u$$

$$= f(t, x_1 + t^{1-m}/(1-m), x_2, \ldots, x_n) \tag{41}$$

The reduction above holds microllocally, but since it holds in every conic neighborhood of any point in T^*M, we can use it at every point of M.

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