数理解析研究所講究録
第 403 巻 1980 年 104-108

REMARKS ON ORDER OF A FUNCTION ON AN ANALYTIC SPACE

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1. Introduction

If f is a germ of an analytic function on an analytic space, we can define two kinds of order of f: algebraic order and analytic order. First, we compare these two. Then we treat a certain analytic map, comparing the order of the pullback with the original order. In both cases, we obtain linear inequalities assuming some conditions on the tangent cone. (In reality, order and inequality are expressed by degree of flatness and inclusion of ideals hereafter.)

2. Terms

Let k denote the field $\mathbb{R}$ or $\mathbb{C}$. If X is an analytic space over k, $(\mathcal{O}_{X,\xi}, m_\xi)$ denotes the local ring of germs of analytic functions at $\xi \in X$.

(a) An element $f \in \mathcal{O}_{X,\xi}$ is called p-flat if $f \in m_\xi^p$.

(b) $f \in \mathcal{O}_{X,\xi}$ is called weakly p-flat if $c \cdot |f(x)| \leq |x - \xi|^p$ holds in a neighbourhood of ξ for some representative $f(\ )$ of f and for some $c > 0$.

(c) Let (X, ξ) be a germ of analytic space and let C and C^* be its tangent cone and projective tangent cone respectively (cf. [W]). C^* is isomorphic to the fiber over ξ of the blowing-up of X with center ξ.

(d) By a curve we mean an irreducible analytic space of dimension 1. A germ of a curve on (X, ξ) has a unique projective tangent in $|C^*|$ (cf. [W]).

(e) Let A be a real analytic set. We can define a d-dimensional Hausdorff (outer) measure $\mu_d: 2^A \longrightarrow [0, \infty]$

on A $(d>0)$. $B\subset A$ is called d-finite if $\mu_d(B)<\infty$ and d-sigmafinite if it is a countable union of d-finite sets.
(f) Let A be a globally irreducible real (resp. complex) analytic set of dimension d. We call $B\subset A$ μ-thin in A if B is (d-1)-sigmafinite (resp. (2d-2)-sigmafinite).
(g) Let $\mathfrak{a}$ be an ideal of $A = k[x_1,\cdots,x_n]$ and let $|C|\subset k^n$ be the intersection of the zero sets of $f\in\mathfrak{a}$. We say that $C = (|C|, A/\mathfrak{a})$ has roperty (Z) if it satisfies the following condition:

If $f\in A$ vanishes on $|C|$, then $f\in\mathfrak{a}$.

If $k = \mathbf{C}$ (resp. $\mathbf{R}$), this property is equivalent to the property $\sqrt{\mathfrak{a}} = \mathfrak{a}$ (resp. ${}^R\sqrt{\mathfrak{a}} = \mathfrak{a}$ (see [Bo-R] for references)).

3. Theorems

By $[R_1]$, (2.2), and Łojasiewicz inequality, we have the following:

Proposition 1. Let $\Phi:(Y,\eta)\longrightarrow(X,\xi)$ be an anlytic map between curves over k. If the induced homomorphism $\varphi: \mathcal{O}_{X,\xi}\longrightarrow\mathcal{O}_{Y,\eta}$ is injective, there exists $\alpha>0$ such that $\varphi^{-1}(m_\eta^p)\subset m_\xi^{[\alpha p]}$ $(p = 0,1,2,\cdots)$. (Here $[\ ]$ denotes the integer part.)

By a modified form of [B-R], (2.19), the above proposition, and by subadditivity of μ_k, we have the following:

Theorem 1. Let (X,ξ) be a germ of a complex space such that its tangent cone has property (Z). Then there exists $\alpha>0$ such that weakly p-flat functions on (X,ξ) are all $[\alpha p]$-flat. In real analytic case, we have only to assume further that the set of the projective tangents of curves is not μ-thin in each component of $|C^*|$.

Remark 1. We may only assume that the functions are weakly p-flat on a subanalytic set which is not μ-thin "directionally".

Remark 2. The real case of this theorem was announced by Risler [R_2] assuming nothing on C and C^*. Later, however, he told me that there was a mistake in his proof ([R_2], Prop. 1).

Let $\Phi: Y \longrightarrow X$ be an analytic map between analytic spaces over k, $\pi: \tilde{X} \longrightarrow X$ the blowing-up with center $\xi \in X$, $\rho: \tilde{Y} \longrightarrow Y$ the blowing-up with center $\Phi^{-1}(\xi)$ and let $\tilde{\Phi}: \tilde{Y} \longrightarrow \tilde{X}$ be the strict transform of Φ (cf. [H]).

Theorem 2. In the above, suppose that the tangent cone C of X has property (Z) and that there exists a subset A in the boundary of $\rho^{-1} \circ \Phi^{-1}(\xi) \subset Y$ such that $\tilde{\Phi}(A)$ is not μ-thin in each component of $|C^*| \subset |\tilde{X}|$. Then there exists $\alpha > 0$ such that, if $(f \circ \Phi)_\eta \in m_\eta^p$ for any $\eta \in \rho(A)$, $f \in m_\xi^{[\alpha p]}$ $(p = 0, 1, 2, \cdots)$.

Remark 3. If $k = \mathbf{C}$, the boundary of $\rho^{-1} \circ \Phi^{-1}(\xi)$ coincides with $\rho^{-1} \circ \Phi^{-1}(\xi)$.

Corollary. Let $\Phi: (Y, \eta) \longrightarrow (X, \xi)$ be a germ of analytic map over k which is open in η (in the sense of [F], p.133). Suppose the same as Theorem 1 for X. Then

(*) there exists $\alpha > 0$ such that $(f \circ \Phi)_\eta \in m_\eta^p$ implies $f \in m_\xi^{[\alpha p]}$ for $p = 0, 1, 2, \cdots$.

(*) implies that the induced homomorphism $\varphi: \mathcal{O}_{X,\xi} \longrightarrow \mathcal{O}_{Y,\eta}$ is injective. The converse is not true. (Considering Samuel functions, we see that (*) implies $\dim \mathcal{O}_{X,\xi} \leq \dim \mathcal{O}_{Y,\eta}$. On the other hand, we have an example of monomorphism φ such that $\dim \mathcal{O}_{X,\xi} > \dim \mathcal{O}_{Y,\eta}$.) Hence the author is interested in the case $r(\varphi) = \dim \mathcal{O}_{X,\xi}$ ($r(\ )$ denotes the geometric rank: cf. [G], [M]). If (X, ξ) is regular, this equality assures (*) (cf. [T], p. 178).

4. C^r functions and completion

By Taylor expansion we can easily see that the results in the previous section are also applicable to C^r functions for

$p = 0, 1, 2, \cdots, r$ (not altering α).

The condition (*) is equivalent to the similar condition on the completion.

Proposition 2. $\varphi: A \longrightarrow B$ be a local homomorphism of local rings and let $\hat{\varphi}: \hat{A} \longrightarrow \hat{B}$ be its completion. Then the following conditions are equivalent.

(i) $\varphi^{-1}(m_B^q) \subset m_A^p$ (ii) $\hat{\varphi}^{-1}(m_{\hat{B}}^q) \subset m_{\hat{A}}^p$

This proposition follows from the fact that $\hat{A}$ (resp. $\hat{B}$) is faithfully flat over A (resp. B).

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