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ON HOLOMORPHIC EQUIVALENCE OF BOUNDED DOMAINS IN COMPLETE KÄHLER MANIFOLDS OF NEGATIVE CURVATURE

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1. Introduction

Suppose D_1 and D_2 are two bounded domains in the complex n-space $\mathbb{C}^n$, $n \geqq 2$, with C^∞ boundaries ∂D_1 and ∂D_2, respectively. One of the fundamental problems in several complex variables is to determine geometric conditions which imply that D_1 and D_2 are biholomorphically equivalent. It has been known from Bochner[1]-Hartogs' theorem that if ∂D_1 and ∂D_2 are CR-diffeomorphic, then D_1 and D_2 are biholomorphic. The same is true even for those domains in a Stein manifold (Shiga[5]).

In this note we are concerned with the problem for domains in complete Kähler manifolds of negative curvature. Our result is stated as follows.

THEOREM. *Let M and N be complete Kähler manifolds of complex dimension $n \geqq 2$. Let $D_1 \subset M$ and $D_2 \subset N$ be relatively compact domains in M and N with C^∞ boundaries ∂D_1 and ∂D_2, respectively. Suppose that (i) there exists a CR-diffeomorphism $f : \partial D_1 \to \partial D_2$ which extends to a homotopy equivalence of D_1 to D_2, (ii) N has adequately negative curvature in the sense of Siu[6], and (iii) the boundary ∂D_2 is convex. Then D_1 and D_2 are biholomorphically equivalent.*

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It should be noted that the curvature hypothesis (ii) is assumed only on the target manifold N. According to Siu[7], the classical bounded symmetric domains with their invariant Kähler metrics, hence their quotient Kähler manifolds also, have adequately negative curvature. The convexity hypothesis (iii), assumed on the boundary ∂D_2 of $D_2 \subset N$, requires that the second fundamental form of ∂D_2 in N with respect to the inward unit normal vector is positive semidefinite everywhere. Hopefully this hypothesis may be weakened.

Some part of our theorem can be seen in Wood[7]. I wish to thank him for making his manuscript available during the preparation of this note.

2. Preliminaries

First we fix some concepts in the theorem. Let $D_1 \subset M$ and $D_2 \subset N$ be as in the theorem. Let J denote the complex structure of M. A C^∞ mapping $f : \partial D_1 \to \partial D_2$ is said to be a CR-<u>mapping</u> if the differential df of f restricted to the complex subspace $T_p(\partial D_1) \cap JT_p(\partial D_1)$ of the real tangent space $T_p(\partial D_1)$ is complex linear at each point $p \in \partial D_1$. Note that $f : \partial D_1 \to \partial D_2$ is a CR-mapping if and only if it satisfies the <u>tangential Cauchy-Riemann equation</u> $\bar{\partial}_b f = 0$, where $\bar{\partial}_b f = \bar{\partial} f \circ \pi$, π being the orthogonal projection $\pi : T_p(D_1) \to T_p(\partial D_1) \cap JT_p(\partial D_1)$ for each $p \in \partial D_1$ (cf. [2]). A CR-<u>diffeomorphism</u> is one for which f and f^{-1} are CR-mappings.

We need the notion of adequate negativity, defined by Siu[6], of the curvature of a Kähler manifold. The curvature tensor of a Kähler n-manfold N is said to be adequately negative at $q \in N$ if the following hold: Let $h : U \to N$ be a C^∞ mapping of an open neighborhood U of $0 \in \mathbb{C}^n$ to N with $h(0) = q$. Let (z^i) denote a local complex coordinates of $\mathbb{C}^n$ around 0 and (w^α) that of N around q. Then the curvature tensor $(R_{\alpha\bar\beta\gamma\bar\delta})$ of N enjoys the properties that (a) $\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar\beta\gamma\bar\delta}\, \xi^{\alpha\bar\beta}_{\bar i\bar j}\, \overline{\xi^{\delta\bar\gamma}_{\bar i\bar j}} \geqq 0$ for all $1 \leqq i,j \leqq n$, where $\xi^{\alpha\bar\beta}_{\bar i\bar j} = (\partial_{\bar i} h^\alpha)(0)\overline{(\partial_j h^\beta)(0)} - (\partial_{\bar j} h^\alpha)(0)\overline{(\partial_i h^\beta)(0)}$, $\partial_{\bar i} h^\alpha = \partial h^\alpha / \partial \overline{z^i}$ etc., and (b) if h is a local diffeomorphism around 0 and $\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar\beta\gamma\bar\delta}\, \xi^{\alpha\bar\beta}_{\bar i\bar j}\, \overline{\xi^{\delta\bar\gamma}_{\bar i\bar j}} = 0$ at q, then either $\partial h = 0$ or $\bar\partial h = 0$ at 0. If the curvature tensor of N is adequately negative everywhere, we simply say that N has adequately negative curvature. The adequate negativity of curvature is stronger than requiring nonpositive sectional curvature. For examples of Kähler manifolds having adequately negative curvature, see Siu[6].

3. Proof of the theorem

Let $D_1 \subset M$ and $D_2 \subset N$ be as in the theorem. By hypothesis (i), we have a CR-diffepmorphism $f : \partial D_1 \to \partial D_2$ which extends to a homotopy equivalence $\tilde f : D_1 \to D_2$, which may be assumed to be C^∞. Since the sectional curvature of N is nonpositive everywhere by hypothesis (ii) and the boundary ∂D_2 of D_2 is assumed to be convex by hypothesis (iii), it then follows from a theorem of Hamilton[3] that there exists a harmonic mapping $h : D_1 \to D_2$ which is homotopic to $\tilde f$

relative to ∂D_1. We refer to Eells-Lemaire[2] for the definition and the fundamental properties of harmonic mappings. Note that h is C^∞ up to the boundary.

In consequence, we may assume that there exists a harmonic homotopy equivalence $h : D_1 \to D_2$ such that $h|\partial D_1 : \partial D \to \partial D_2$ is a CR-diffeomorphism. We are going to prove that h is a desired biholomorphic equivalence of D_1 to D_2.

<u>Assertion</u> 1. h is holomorphic on D_1.

Let g and ω denote the Kähler metric and the Kähler form of N, respectively. Let (z^i) and (w^α) denote respectively the local complex coordinates of M and N, and let $(R_{\alpha\bar{\beta}\gamma\bar{\delta}})$ denote the curvature tensor of N. Denote by $<\ ,\ >$ contraction of tensors and consider the (1,1)-form $<g,\ \bar{\partial}h \wedge \partial\bar{h}>$ on D_1 defined in terms of local coordinates by

$$<g,\ \bar{\partial}h \wedge \partial\bar{h}> = \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\ \bar{\partial}h^\alpha \wedge \partial\overline{h^\beta}\ .$$

It is then known in Siu[6] that by harmonicity of h, at all points $p \in D_1$ we have

$$(1) \qquad d\{\bar{\partial}<g,\ \bar{\partial}h \wedge \partial\bar{h}> \wedge \omega^{n-2}\} = \partial\bar{\partial}<g,\ \bar{\partial}h \wedge \partial\bar{h}> \wedge \omega^{n-2} = \sigma\omega^n - \chi\omega^n\ ,$$

where, with respect to a local complex coordinates orthonormal at p,

$$(2) \qquad \sigma = \frac{1}{n(n-1)} \sum_{\substack{\alpha,\beta,\gamma,\delta \\ 1 \leqq i<j \leqq n}} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\ \xi^{\alpha\bar{\beta}}_{\bar{i}\bar{j}}\ \overline{\xi^{\delta\bar{\gamma}}_{\bar{i}\bar{j}}}\ ,$$

$\xi^{\alpha\bar{\beta}}_{\bar{i}\bar{j}} = \partial_{\bar{i}}h^\alpha \cdot \overline{\partial_j h^\beta} - \partial_{\bar{j}}h^\alpha \cdot \overline{\partial_i h^\beta}$, and χ is some nonpositive function on D_1. Note that the adequate negativity of the curvature of N implies that $\sigma \geqq 0$.

On the other hand, at each point $p \in \partial D_1$ we have

(3) $$\bar{\partial}\langle g, \bar{\partial}h \wedge \partial\bar{h}\rangle \wedge \omega^{n-2} = -\langle g, \bar{\partial}h \wedge \bar{D}\partial_b\bar{h}\rangle \wedge \omega^{n-2} .$$

Here $\bar{D}$ denotes covariant $\bar{\partial}$ exterior differentiation of $h^*TN\otimes\mathbb{C}$-valued forms on M, which in terms of local coordinates is defined to be $\bar{D}\overline{\partial h^\beta} = \bar{\partial}\overline{\partial h^\beta} - \sum_{\alpha,\gamma} \overline{\Gamma^\beta_{\alpha\gamma}}\, \overline{\bar{\partial}h^\alpha} \wedge \overline{\partial h^\gamma}$, $\Gamma^\beta_{\alpha\gamma}$ being the Christoffel symbols of N. $\bar{\partial}_b$ denotes the tangential Cauchy-Riemann operator and $\partial_b\bar{h} = \bar{\partial}_b h$. The proof of (3) is done by a straightforward calculation (cf. [7]). Note that $\partial_b\bar{h} = 0$, because $h|\partial D_1$ is a CR-mapping. Hence we have

(4) $$\bar{\partial}\langle g, \bar{\partial}h \wedge \partial\bar{h}\rangle \wedge \omega^{n-2} = 0 \quad \text{on} \quad D_1.$$

Now we integrate (1) over D_1. Then it follows from Stokes' theorem and (4) that

$$\int_{D_1} (\sigma\omega^n - \chi\omega^n) = 0 ,$$

from which we obtain $\sigma \equiv 0$ and $\chi \equiv 0$, for $\sigma \geqq 0$ and $\chi \leqq 0$ on D_1. As a result, we get from (2) that for all $1 \leqq i,j \leqq n$

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\, \xi^{\alpha\bar{\beta}}_{\bar{i}\bar{j}}\, \overline{\xi^{\delta\bar{\gamma}}_{\bar{i}\bar{j}}} = 0 \quad \text{on} \quad D_1.$$

Recall that h is a local diffeomorphism near ∂D_1. Then the adequate negativity of the curvature of N implies that $\partial h = 0$ or $\bar{\partial}h = 0$ at each point near ∂D_1. Since h is a harmonic mapping, it then follows as in Siu[6] from the unique continuation property that $\partial h \equiv 0$ on D_1 or $\bar{\partial}h \equiv 0$ on D_1. But $\bar{\partial}_b h = 0$ on ∂D_1 and the rank of $dh|\partial D_1$ is 2n-1, so

$\partial h \equiv 0$ is impossible. Hence we conclude that $\bar{\partial} h \equiv 0$ on D_1, that is, h is holomorphic on D_1.

Assertion 2. h is a biholomorphic mapping of D_1 to D_2.

Let V be the set of points of D_1 where h is not locally diffeomorphic. V is a compact complex-analytic subvariety in D_1 of pure complex codimension 1, for locally V is defined by $\det(\partial w^\alpha / \partial z^i)$ and h is locally diffeomorphic near ∂D_1. Note that h is of degree 1 and hence maps $D_1 - h^{-1}(h(V))$ bijectively onto $h(D_1) - h(V)$. Thus it suffices to prove that V is empty. Assume the contrary, namely assume that $V \neq \phi$. Then V defines a nonzero homology class $[V]$ in $H_{2n-2}(D_1 ; \mathbb{R})$. Since h is a proper mapping, it follows from a theorem of Remmert[4] that $h(V)$ is a compact complex-analytic subvariety of complex codimension at least 2. Hence $[V]$ in $H_{2n-2}(D_1 ; \mathbb{R})$ is mapped by h to the zero element in $H_{2n-2}(D_2 ; \mathbb{R})$, that is, $h_*([V]) = 0$ in $H_{2n-2}(D_2 ; \mathbb{R})$, contradicting the fact that h is a homotopy equivalence of D_1 to D_2.

The proof of the theorem is now complete.

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