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Invariant Measures for Homeomorphisms with Weak Specification

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§ 0 INTRODUCTION.

In this paper one considers the space of measures provided with the weak topology. In [7,8] , K. Sigmund discussed some categories in the space of invariant measures for homeomorphisms satisfying specification. The ingredient of his proofs is in the densely periodic property of homeomorphisms with specification. It is known that weak specification for homeomorphisms is strictly weaker than specification. For example, N. Aoki proves in [2] that there exist group automorphisms without densely periodic property.

Our aim is to show that the results of K. Sigmund hold for homeomorphisms satisfying weak specification (Theorems 1 and 3) . The idea of the proof is in constructing the property "smallest sets" (see § 2) that is found in the weak specification property.

§ 1 MAIN RESULTS.

Let X be a compact metric space with metric d and $\mathfrak{M}(X)$ be the space of Borel probability measures of X with metric $\bar{d}$ where $\bar{d}$ is defined by

$$\bar{d}(\mu,\nu) = \inf\{\varepsilon\ ;\ \mu(B) \leq \nu(\{x\in X\ ;\ d(x,B) \leq \varepsilon\}) + \varepsilon \text{ and }$$
$$\nu(B) \leq \mu(\{x\in X\ ;\ d(x,B) \leq \varepsilon\}) + \varepsilon \text{ for all Borel sets } B\}$$

([5] p.9 or [3] p.238) .

Define a point measure $\delta(x)$ by $\delta(x)(B) = 1$ if $x \in B$ and $\delta(x)(B) = 0$ if $x \notin B$ (Borel set B) , and denote by $B(X, \varepsilon)$ an ε-closed ball about x in X. For arbtrary finite sets $\{x_1, \dots x_n\}$ and $\{\mu_i \in \mathfrak{M}(X);\ 1 \leq i \leq n\}$ with $\operatorname{card}\{1 \leq i \leq n\ ;\ \mu_i(B(x_i, \varepsilon)) < 1\}/n < \varepsilon$, we get easily $\bar{d}(\frac{1}{n}\sum_{i=1}^{n}\delta(x_i)\ ,\ \frac{1}{n}\sum_{i=1}^{n}\mu_i) < \varepsilon$. It is clear that the map $x \longrightarrow \delta(x)$ is a homeomorphism from X onto a subset of $\mathfrak{M}(X)$.

Let σ be a self-homeomorphism of X . Then σ induces a homeomorphism $\sigma : \mathfrak{M}(X) \longrightarrow \mathfrak{M}(X)$ by $\sigma\mu(B) = \mu(\sigma^{-1}B)$ (Borel set B and $\mu \in \mathfrak{M}(X)$) such that $\delta(\sigma x) = \sigma\delta(x)$. Hence we can consider that (X, σ) is a subsystem of $(\mathfrak{M}(X), \sigma)$. It is known ([5] , p.17) that the set $\mathfrak{M}_\sigma(X)$ of σ-invariant measures is a compact convex set.

Let $\mathcal{E}(X)$ denote the set of ergodic measures in $\mathfrak{M}(X)$. Then $\mathcal{E}(X)$ is a nonempty G_δ-set in $\mathfrak{M}_\sigma(X)$ ([5] , p.25) . Let $\mathcal{S}(X)$ denote the set of strongly mixing measures in $\mathfrak{M}_\sigma(X)$, $\mathcal{L}(X)$ denote the set of measures positive on all nonempty open sets in $\mathfrak{M}_\sigma(X)$, and $\mathcal{N}(X)$ denote the set of non-atomic measures in $\mathfrak{M}_\sigma(X)$. We denote by $V_\sigma(x)$ the set of ω-limits of the sequence $\{\frac{1}{n}\sum_{j=0}^{n-1}\delta(\sigma^j x)\}_{n=1}^{\infty}$ for $x \in X$. Then we know ([5] , p.18) that for every $x \in X$, $V_\sigma(x)$ is a nonempty compact connected subsets of $\mathfrak{M}_\sigma(X)$.

Let X and σ be as above. Then (X, σ) is said to satisfy <u>weak specification</u> if for $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that for every $k \geq 1$, k points $x_1, \dots, x_k \in X$ and for every set of integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$ with $a_i - b_i \geq M(\varepsilon)$ ($2 \leq i \leq k$) , the set $\hat{B} = \bigcap_{i=1}^{k}\bigcap_{j=a_i}^{b_i}\sigma^{-j}B(\sigma^j x_i, \varepsilon)$ is nonempty. Since

$$\emptyset \neq \bigcap_{r=1}^{\infty}\bigcap_{n=-r}^{r}\bigcap_{i=1}^{k}\bigcap_{j=a_i+nq}^{b_i+nq}\sigma^{-j}B(\sigma^{j-nq}x_i, \varepsilon) \subset \hat{B} \text{ for all } q \geq b_k - a_1 + M(\varepsilon),$$

we get easily that $\hat{B}$ contains a σ^q-invariant subset. When (X, σ)

obeys weak specification and has the following additional condition; for every $q \geqslant b_k - a_1 + M(\varepsilon)$, there is an $x \in B$ with $\sigma^q x = x$, we say (X, σ) to satisfy <u>specification</u>.

<u>THEOREM 1.</u> Let X be a compact metric space ($\mathrm{card}(X) > 1$) , and σ be a self-homeomorphism of X . If (X, σ) satisfies weak specification, then $\mathcal{E}(X)$, $\mathcal{S}(X)$, and $\mathcal{N}(X)$ are dense G_δ-sets of $\mathfrak{M}_\sigma(X)$, and $\mathcal{S}(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.

<u>THEOREM 2.</u> Let X and σ be as in Theorem 1. If (X, σ) satisfies weak specification, then $(\mathfrak{M}(X), \sigma)$ has the specification property.

<u>THEOREM 3.</u> Let X and σ be as in Theorem 1. If (X, σ) satisfies weak specification, then for every nonempty compact connected subset V of $\mathfrak{M}_\sigma(X)$, there is an $x \in X$ such that $V_\sigma r(x) = V$ for all $r \geqslant 1$ and the set of such points x is a dense set in X .

§ 2 <u>AUXILIARY RESULTS.</u>

In this section we show two result which are used in the proof of the theorems. Hereafter X is a compact metric space with metric d and σ is a self-homeomorphism of X .

A nonempty closed subset Δ is said to be a <u>smallest set</u> if there is an integer $q \geqslant 1$ such that $\sigma^q \Delta = \Delta$ and Δ contains no completely σ^q-invariant closed subsets. We call the least positive integer in the set of such $q \geqslant 1$ the <u>period</u> of Δ , and we denote it by $\mathrm{per}(\Delta)$. Obviously, $\sigma^i \Delta \cap \Delta = \emptyset$ for i with $1 \leqslant i < \mathrm{per}(\Delta) - 1$. Let Δ be a smallest set. Then $\tilde{\Delta} = \bigcup_{i=0}^{\mathrm{per}(\Delta)-1} \sigma^i \Delta$ is a minimal sets under σ ; i.e. Δ contains no completely σ-invariant closed proper

subsets. Since $\tilde{\Delta}$ is compact and $\sigma\tilde{\Delta} = \tilde{\Delta}$, as before we can consider the space $\mathcal{M}_\sigma(\tilde{\Delta})$ of σ-invariant Borel probability measures of $\tilde{\Delta}$. Then every $\mu \in \mathcal{M}_\sigma(\tilde{\Delta})$ defines a measure $\bar{\mu} \in \mathcal{M}_\sigma(X)$ by $\bar{\mu}(B) = \mu(B \cap \tilde{\Delta})$ for Borel sets B of X. It is clear that if $\mu \in \mathcal{M}_\sigma(\tilde{\Delta})$ is ergodic, then $\bar{\mu} \in \mathcal{E}(X)$. We remark that $\mu(\sigma^j\Delta) = 1/\mathrm{per}(\Delta)$ $(0 \leq j < \mathrm{per}(\Delta))$ for all $\mu \in \mathcal{M}_\sigma(\tilde{\Delta})$. Define $\bar{\mu}_j \in \mathcal{M}(X)$ $(0 \leq j)$ by $\bar{\mu}_j(B) = \mathrm{per}(\Delta)\bar{\mu}(B \cap \sigma^j\Delta)$ for Borel sets B of X.
Then we have $\bar{\mu} = \frac{1}{\mathrm{per}(\Delta)}\sum_{j=0}^{\mathrm{per}(\Delta)-1}\bar{\mu}_j$. We say that $x \in X$ is a <u>generic point</u> for $\mu \in \mathcal{M}_\sigma(X)$ if $V_\sigma(x) = \{\mu\}$. Every $\mu \in \mathcal{E}(X)$ has generic points and μ-invariant measure of the set of such points is one (c.f. see [5], p.25).

<u>PROPOSITION 1.</u> If (X,σ) satisfies weak specification, then $\mathcal{E}(X)$ is dense in $\mathcal{M}_\sigma(X)$.

<u>PROOF.</u> It is clear that $\mathcal{E}(X) \neq \emptyset$. First we prove that for every $\mu_1, \mu_2 \in \mathcal{E}(X)$, every $t \in [0,1]$ and every $\varepsilon > 0$, there exists $\nu \in \mathcal{E}(X)$ with $\bar{d}(\nu, t\mu_1 + (1-t)\mu_2) < \varepsilon$.

Take an integer $m > 4/\varepsilon$, then there exists an integer m_1 with $1 \leq m_1 \leq m-1$ such that $\left|\frac{m_1}{m} - t\right| \leq \frac{1}{m}$. It follows from the definition of $\bar{d}$ that

$$\bar{d}\left(t\mu_1 + (1-t)\mu_2, \frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2\right) < \varepsilon/2 .$$

Let x_1 and x_2 be generic points for μ_1 and μ_2, respectively, and choose $M = M(\varepsilon/4)$ as in the definition of weak specification. Since x_i is a generic point for μ_i $(i = 1,2)$ we can find an $N_0 > 4M/\varepsilon$ such that for all $n \geq N_0$, $\bar{d}(\mu_i, \frac{1}{n}\sum_{j=0}^{n-1}\delta(\sigma^j x_i)) < \varepsilon/4$ $(i = 1,2)$.

Put $N_1 = m_1N_0 - M$ and $N_2 = (m - m_1)N_0 - M$. Then we can calculate easily that

$$\bar{d}\Big(\frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2 , (N_1 + N_2 + 2M)^{-1}\sum_{i=1}^{N_i+M-1} \delta(\sigma^j x_i)\Big)$$

$$\leq \bar{d}\Big(\frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2 , \frac{m_1}{m}\Big(\frac{1}{N_1+M}\sum_{j=o}^{N_1+M-1} \delta(\sigma^j x_1)\Big)$$

$$+ \frac{m-m_1}{m}\Big(\frac{1}{N_2+M}\sum_{j=o}^{N_2+M-1} \delta(\sigma^j x_2)\Big)\Big) < \varepsilon/4 .$$

To use the weak specification property we put $a_1 = 0$, $b_1 = N_1$, $a_2 = b_1 + M$, $b_2 = a_2 + N_2$, $q = b_2 + M$, $y_1 = x_1$ and $y_2 = \sigma^{-a_2} x_2$. Since X is compact, it follows that there is a smallest set Λ such that $\sigma^q \Lambda = \Lambda \subset \bigcap_{i=1}^{2} \bigcap_{j=a_i}^{b_i} \sigma^{-j}B(\sigma^j y_i , \varepsilon/4)$. Since every ergodic measure $\nu \in \mathcal{M}_\sigma(\tilde{\Lambda})$ satisfies $\bar{\nu}_j(B(\sigma^j y_i , \varepsilon/4)) = 1$ ($a_i \leq j \leq b_i$, $i = 1, 2$), we have $\mathrm{card}\{0 \leq j \leq q-1 ; \bar{\nu}_j(B(\sigma^j y_i , \varepsilon/4) < 1\}/q < 2M/q < \varepsilon/4$, We remark that $\bar{\nu} = \frac{1}{q}\sum_{j=o}^{q-1}\bar{\nu}_j$ since q is devided by $\mathrm{per}(\Lambda)$. Then

$$\bar{d}\Big(\bar{\nu} , \frac{1}{q}\Big(\sum_{i=1}^{2}\sum_{j=o}^{N_i+M-1} \delta(\sigma^j x_i)\Big)\Big)$$

$$= \bar{d}\Big(\frac{1}{q}\sum_{j=o}^{q-1}\bar{\nu}_j , \frac{1}{q}\sum_{i=1}^{2}\sum_{j=a_i}^{b_i+M-1} \delta(\sigma^j y_i)\Big) \leq \varepsilon/4 .$$

Hence

$$\bar{d}(\bar{\nu}, t\mu_1 + (1-t)\mu_2) \leq \bar{d}\Big(\bar{\nu}, \frac{1}{q}\sum_{i=1}^{2}\sum_{j=o}^{N_i+M-1} \delta(\sigma^j x_i)\Big)$$

$$+ \bar{d}\Big(\frac{1}{q}\sum_{i=1}^{2}\sum_{j=o}^{N_i+M-1} \delta(\sigma^j x_i) , \frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2\Big) \quad (\text{ since } q = N_1 + N_2 + 2M)$$

$$+ \bar{d}\left(\frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2 , t\mu_1 + (1-t)\mu_2\right) < \varepsilon .$$

We use induction to get the conclusion. Take $\mu \in \mathcal{M}_\sigma(X)$, then for every $\varepsilon > 0$ there exist a $k \geq 1$, $\mu_1, \dots, \mu_k \in \mathcal{E}(X)$ and $t_1, \dots, t_k \geq 0$ with $t_1 + t_2 + \dots + t_k = 1$ such that $\bar{d}(\mu, \sum_{i=1}^{k} t_i\mu_i) < \varepsilon/2$ ([5] , p.25) . By the first part of the proof, there is a $\nu_1 \in \mathcal{E}(X)$ such that $\bar{d}(t_1/(t_1+t_2)\mu_1 + t_2/(t_1+t_2)\mu_2 , \nu_1) < \varepsilon/4$. Also there is a $\nu_2 \in \mathcal{E}(X)$ such that $\bar{d}((t_1+t_2)/(t_1+t_2+t_3)\nu_1 + t_3/(t_1+t_2+t_3)\mu_3 , \nu_2) < \varepsilon/8$. Put $t^{(i)} = \sum_{j=1}^{i} t_j$ for $1 \leq i \leq k$, then by the definition of $\bar{d}$ that

$$\bar{d}\left(\sum_{j=1}^{3} \frac{t_j}{t^{(3)}}\mu_j , \nu_2\right)$$

$$\leq \bar{d}\left(\frac{t^{(2)}}{t^{(3)}}\left(\frac{t_1}{t^{(2)}}\mu_1 + \frac{t_2}{t^{(2)}}\mu_2\right) + \frac{t_3}{t^{(3)}}\mu_3 , \frac{t^{(2)}}{t^{(3)}}\nu_1 + \frac{t_3}{t^{(3)}}\mu_3\right)$$

$$+ \bar{d}\left(\frac{t^{(2)}}{t^{(3)}}\nu_1 + \frac{t_3}{t^{(3)}}\mu_3 , \nu_2\right)$$

$$< \varepsilon/4 + \varepsilon/8 .$$

When $\nu_i \in \mathcal{E}(X)$ $(2 \leq i \leq k-2)$ is already defined, by the above way we can find $\nu_{i+1} \in \mathcal{E}(X)$ such that

$$\bar{d}\left(\frac{t^{(i+1)}}{t^{(i+2)}}\nu_i + \frac{t_{i+2}}{t^{(i+2)}}\mu_{i+1} , \nu_{i+1}\right) < \varepsilon/2^{i+1}.$$

Since $\nu_{k-1} \in \mathcal{E}(X)$ and $\bar{d}(\sum_{i=1}^{k} t_i\mu_i , \nu_{k-1}) \leq \sum_{i=1}^{k-1} \frac{1}{2^{i+1}} < \varepsilon/2$, the proof is completed.

Let us put $Z(\Delta, \delta) = \{0 \leq j < \mathrm{per}(\Delta) ; \ \mathrm{diam}(\sigma^j\Delta) < \delta\}$ for a smallest set Δ and $\delta > 0$. Denote by $A(\delta)$ the collection of smallest sets Λ with prime period satisfying the conditions ;

$$\mathrm{per}(\Lambda) > \delta^{-1} \quad \text{and} \quad \mathrm{card}(Z(\Lambda,\delta))/\mathrm{per}(\Lambda) > 1-\delta .$$

It is easy to checked that $A(\delta_1) \subset A(\delta_2)$ when $\delta_1 < \delta_2$.

PROPOSITION 2. If (X, σ) ($\mathrm{card}(X) > 1$) satisfies weak specificatio for every $\delta > 0$ with $\delta < \mathrm{diam}(X)/4$ and for every $\mu \in \mathcal{M}_\sigma(X)$ there exists a $\Lambda \in A(\delta)$ such that every measure ν in $\mathcal{M}_\sigma(\tilde{\Lambda})$ holds $\bar{d}(\mu, \bar{\nu}) < \delta$.

PROOF. Since $\mathcal{E}(X)$ is dense in $\mathcal{M}_\sigma(X)$ by Proposition 1, there is an $\mu_1 \in \mathcal{E}(X)$ such that $\bar{d}(\mu,\mu_1) < \delta/3$. Choose $M = M(\delta/3)$ as in the definition of weak specification. Let x_1 be a generic point for μ_1. Then there is an $N_0 > 6M/\delta$ such that $\bar{d}(\frac{1}{n}\sum_{j=0}^{n-1}\delta(\sigma^j x_1),\mu_1) < \delta/3$ ($n \geq N_0$) . Take a prime p with $p > N_0 + 2M$ and put $N = p - 2M$. For $x_2 \in X$ with $d(\sigma^{N+M}x_2, x_1) > 2\delta$, putting $a_1 = 0$, $b_1 = N$ and $a_2 = b_2 = N + M$. As before we have that there is a smallest set Λ such that $\sigma^p\Lambda = \Lambda \subset \bigcap_{i=1}^{2}\bigcap_{j=a_i}^{b_i}\sigma^{-j}B(\sigma^j x, \delta/3)$.

Since $\Lambda \cap \sigma^{N+M}\Lambda \subset B(x_1, \delta/3) \cap B(\sigma^{N+M}x_2, \delta/3) = \phi$, we get $\mathrm{per}(\Lambda) \neq 1$ and $\mathrm{per}(\Lambda)$ devides p. But p is prime so that $\mathrm{per}(\Lambda) = p > \delta^{-1}$. Since $\{0, 1, \dots, N\} \subset Z(\Lambda, \delta)$ and $\mathrm{card}(Z(\Lambda,\delta))/p \geq 1 - \frac{2M}{p} \quad 1 - \frac{\delta}{3}$, we get $\Lambda \in A(\delta)$. Since $\bar{\nu}_j(B(\sigma^j x_1, \delta/3)) = 1$ ($0 \leq j \leq N$) for all $\nu \in \mathcal{M}_\sigma(\tilde{\Lambda})$, it follows that

$$\mathrm{card}\{0 \leq j < p \; ; \; \bar{\nu}_j(B(\sigma^j x_1, \delta/3)) < 1\} < \frac{p-(N+1)}{p} < 2M/p < \delta/3 .$$

Since $\bar{\nu} = \frac{1}{p}\sum_{j=0}^{p-1}\bar{\nu}_j$, we get easily that $\bar{d}(\frac{1}{p}\sum_{j=0}^{p-1}\delta(\sigma^j x_1), \bar{\nu}) = \bar{d}(\frac{1}{p}\sum_{j=0}^{p-1}\delta(\sigma^j x_1), \frac{1}{p}\sum_{j=0}^{p-1}\nu_j) < \delta/3$. We now have

$$\bar{d}(\mu_1, \bar{\nu}) \leq \bar{d}(\mu_1, \frac{1}{p}\sum_{j=o}^{p-1}\delta(\sigma^j x_1)) + \bar{d}(\frac{1}{p}\sum_{j=o}^{p-1}\delta(\sigma^j x_1), \bar{\nu}) < 2\delta/3$$

$$(\nu \in \mathcal{M}_\sigma(\tilde{\Delta}))$$

and the proof is completed.

3 PROOF OF THEOREMS.

In this section we prove Theorems 1, 2 and 3 that are mentioned in § 1 .

PROOF OF THEOREM 1. Since $\mathcal{E}(X)$ is dense in $\mathcal{M}_\sigma(X)$ by Proposition 1 , $\mathcal{E}(X)$ is a dense G_δ-subset of $\mathcal{M}_\sigma(X)$. Let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be a countable open base of X . Since (X, σ) satisfies weak specification , we can find a smallest set Δ_i with $\Delta_i \subset U_i$ for $U_i \in \mathcal{U}$. For every $i \geq 1$, take $\mu_i \in \mathcal{M}_\sigma(\Delta_i)$, then $\bar{\mu}_i(U_i) \geq \mathrm{per}(\Delta i)^{-1} > 0$. Hence $\mu = \sum_{i=1}^{\infty} \frac{1}{2^i}\mu_i$ is a measure positive on all nonempty **open sets** ; i.e. $\mu \in \mathcal{D}(X)$. It follows from Proposition 21.11 of [5] that $\mathcal{D}(X)$ is a dense G_δ-subset of $\mathcal{M}_\sigma(X)$ unless $\mathcal{D}(X)$ is an empty set. For every integer $r > 0$, $N_r = \{\mu \in \mathcal{M}_\sigma(X) ; \mu(x) < \frac{1}{r}$ for all $x \in X\}$ open in $\mathcal{M}_\sigma(X)$. Using Proposition 2 , we have that N_r is dense in $\mathcal{M}_\sigma(X)$ for all $r \geq 1$. Since $\mathcal{N}(X) = \bigcap_{r=1} N_r$, $\mathcal{N}(X)$ is a dense G_δ-subset of $\mathcal{M}_\sigma(X)$.

Since $\mathcal{D}(X)$ is a dence G_δ-subset of $\mathcal{M}_\sigma(X)$, it is enough to show that $\mathcal{S}(X) \cap \mathcal{D}(X)$ is a set of first category in $\mathcal{M}_\sigma(X)$.

Since $\mathrm{card}(X) > 1$, there is two nonempty disjoint closed neighborhood F_1 and F_2 in X . For $n \geq 2$, put $S(n) = \{\mu \in \mathcal{S}(X) ; \mu(F_1) \geq \frac{1}{n}$ and $\mu(F_2) \geq \frac{1}{n}\}$, then $\mathcal{S}(X) \cap \mathcal{D}(X) = \bigcup_{n=2}^{\infty} S(n)$. Let V_m be the $\frac{1}{m}$-open neighborhood of F_1 for every $m \geq 1$, then $S(n) \subset \bigcup_{m=1} \bigcup_{r=1} E[m,r]$ where $E[m,r] = \bigcap_{j=r}^{\infty} \{\mu \in \mathcal{M}_\sigma(X) ; \mu(V_m \cap \sigma^j V_m) - \mu(F_1)^2 \leq 1/2n^2,$

$\mu(F_1) \geq \frac{1}{n}, \quad \mu(F_2) \geq \frac{1}{n}\}.$

Since V_m $(m \geq 1)$ is open and F_1 and F_2 are closed, it is easy to check that each $E[m,r]$ are closed

We show that for every $m \geq 1$ and $r \geq 1$, $E[m,r]$ is a nowhere dense subset of $\mathcal{M}_\sigma(X)$. For fixed m, take $n \geq 2$ such that $m \leq 2n^2$. For every $\Delta \in A(1/2n^2)$, define a set $Z = \{0 \leq j < per(\Delta)\ ;\ \sigma^j\Delta \cap F_1 \neq \phi$ and $\sigma^j\Delta \not\subset V_m\}$. Then, by the definition of $A(1/2n^2)$, we have $card(Z)/per(\Delta) < 1/2n^2$. For every $\nu \in \mathcal{M}_\sigma(\tilde{\Delta})$, $\bar{\nu}(V_m \cap \sigma^{j\,per(\Delta)}V_m)$ $> \bar{\nu}(F_1) - \frac{1}{2n^2}$ $(j \geq 1)$, and so $\bar{\nu}(V_m \cap \sigma^{j\,per(\Delta)}V_m) - \bar{\nu}(F_1)^2$ $> \bar{\nu}(F_1)(1 - \bar{\nu}(F_1)) - \frac{1}{2n^2}$. This shows that $\bar{\nu} \in E[m,r]$. By Proposition 2, $\bigcup_{\Delta \in A(1/2n^2)} \{\bar{\nu} \in \mathcal{M}_\sigma(X)\ ;\ \nu \in \mathcal{M}_\sigma(\tilde{\Delta})$ is dense in $\mathcal{M}_\sigma(X)\}$.

Hence $\mathcal{S}(X) \cap \mathcal{D}(X)$ is contained in a countable union of nowhere dense closed sets, and so $\mathcal{S}(X) \cap \mathcal{D}(X)$ is a set of first category in $\mathcal{M}_\sigma(X)$.

PROOF OF THEOREM 2 Let $\varepsilon > 0$ be given and $M(\varepsilon/2)$ be as in the definition of weak specification. Let $\mu_1, \ldots, \mu_k \in \mathcal{M}(X)$ be given, as well as integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ and q with $a_i - b_{i-1} \geq M(\varepsilon/2)$ and $q \geq M(\varepsilon/2) + b_k - a_1$. Since $\sigma ; \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$ is uniformly continuous, there exists an $\eta > 0$ such that $\bar{d}(\mu,\nu) < \eta$ implies $\bar{d}(\sigma^j\mu, \sigma^j\nu) > \varepsilon/2$ for $a_1 \leq j \leq b_k$. For some integer $n \geq o$ and $x_r^i \in X$ $(1 \leq r \leq n,\ 1 \leq i \leq k)$ such that putting $\nu_i = \frac{1}{n}\sum_{r=1}^{n} \delta(x_r^i)$ $(1 \leq i \leq k)$, $\bar{d}(\mu_i, \nu_i) < \eta$ for $1 < i \leq k$ (c.f. [5] , p.11) . Since $\sigma : X \longrightarrow X$ satisfies weak specification. there exist a smallest set Δ with $\sigma^q\Delta = \Delta$ and $\Delta \subset \bigcap_{i=1}^{k} \bigcap_{j=a_i}^{b_i} \sigma^{-j}B(\sigma^j x_r^i, \varepsilon/2)$ for $1 \leq r \leq n$. Take $\rho^r \in \mathcal{M}_\sigma(\tilde{\Delta})$, and put

$= \frac{1}{n}\sum_{r=1}^{n}\bar{\rho}^{r}{}_{o}$ where $\bar{\rho}^{r}{}_{o}(B) = \mathrm{per}(\Delta)\bar{\rho}^{r}(B \cap \Delta)$ for Borel set B .

Obviously $\sigma^{q}\rho = \rho$. Also $\bar{d}(\sigma^{j}\rho, \sigma^{j}\nu_{i})$

$= \bar{d}(\frac{1}{n}\sum_{r=1}^{n}\sigma^{j}\rho^{r}, \frac{1}{n}\sum_{r=1}^{n}\delta(\sigma^{j}x_{r}^{i})) \leq \varepsilon/2$ and hence $\bar{d}(\sigma^{j}\rho, \sigma^{j}\mu_{i})$

$\leq \varepsilon$ for $a_{i} \leq j \leq b_{i}$, $i = 1, \ldots, k$. Hence $\sigma : \mathcal{M}(X) \to \mathcal{M}(X)$

satisfies specification.

<u>PROOF OF THEOREM 3.</u> Since V is compact and connected, by Proposition 2 there exist a sequence $\{\varepsilon_{n}\}_{n=1}$ of positive number with $\varepsilon_{n} \searrow 0$ and a sequence $\{\Delta_{n}\}_{n=1}$ in $A(\varepsilon_{n})$ such that for some $\mu_{n} \in \mathcal{M}_{\sigma}(\tilde{\Delta}_{n})$ the followings hold;

(a) $B_{n} \cap B_{n+1} \cap V \neq \emptyset$,

(b) $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} B_{n} = V$,

where B_{n} ($n \geq 1$) is the ε_{n}-closed neighborhood of $\bar{\mu}_{n}$ in $\mathcal{M}(X)$.

We have to show that for every $x_{o} \in X$ and $\delta > 0$ there exists an $x \in B(x_{o}, \delta)$ such that $V_{\sigma^{r}}(x) = V$ for all $r \geq 1$. For every $n \geq 1$, take an $x_{n} \in \Delta_{n}$. Since (X, σ) satisfies weak specification, there exist positive integers M_{n} ($n \geq 0$) such that for every set of integers $a_{o} \leq b_{o} < a_{1} \leq b_{1} < a_{2} \leq b_{2} < \cdots$ with $a_{n} - b_{n-1} \geq M_{n-1}$ ($n \geq 1$) . We know that there exist an $x \in X$ such that $d(\sigma^{j}x, \sigma^{j}x_{n}) \leq \varepsilon_{n}$ ($a_{n} \leq j \leq b_{n}$, $n \geq 0$) and $d(\sigma^{j}x, \sigma^{j}x_{o}) \leq \delta$ ($a_{o} \leq j \leq b_{o}$) (c.f. see Orbit specification lemma in [8]) . With the above notations, take a_{n} and b_{n} ($n \geq 0$) as follows;

(i) $a_{o} = b_{o} = 0$,

(ii) a_n is devided by $n!$ and

$b_{n-1} + M_{n-1} \leq a_n < b_{n-1} + M_{n-1} + n!$ $(n \geq 1)$ and

(iii) $b_n = a_n + (n+1)!\,(a_n + M_n)\,\mathrm{per}(\Delta_n)\,\mathrm{per}(\Delta_{n-1})$ $(n \geq 1)$.

Then, we have an $x \in B(x_o, \delta)$ with $d(\sigma^j x, \sigma^j x_n) \leq \varepsilon_n$ $(a_n \leq j \leq b_n$, $n \geq 1)$.

We have to show that $V_\sigma r(x) = V$ for all $r \geq 1$. Though the proof is similar to that in [8] , we sketch it for completeness.

It is clear that for $r \geq 1$ there is $N_0 > r$ such that $\mathrm{per}(\Delta_n) > r$ for all $n \geq N_0$. Now we fix the integer r, n with $n \geq N_0$ and k with $b_n/r < k \leq b_{n+1}/r$, and write

$$A_1 = A \cap \left[\frac{a_n}{r}, \frac{b_n}{r}\right)$$

where $A = \{0 \leq j \leq k \ ; \ j \text{ is an integer}\}$.
Take k' with $k - \mathrm{per}(\Delta_{n+1}) < k' \leq k$ such that $k' - \frac{a_{n+1}}{r}$ is devided by $\mathrm{per}(\Delta_{n+1})$.

Then it is easy to see that $A_2 = A \cap \left[\frac{a_{n+1}}{r}, k'\right)$ is nonempty when $k \geq \frac{a_{n+1}}{r} + \mathrm{per}(\Delta_{n+1})$ and A_2 is empty when $k < \frac{a_{n+1}}{r} + \mathrm{per}(\Delta_{n+1})$. Obviously $\mathrm{per}(\Delta_{n+1})$ devides $\mathrm{card}(A_2)$. By (iii) , $\mathrm{per}(\Delta_n)$ devides $\mathrm{card}(A_1)$. Remark that $\mathrm{per}(\Delta_n)$ and $\mathrm{per}(\Delta_{n+1})$ are prime numbers. Since $n \geq N_0$, $\mathrm{per}(\Delta_n)$ and $\mathrm{per}(\Delta_{n+1})$ are both prime to the integer r , so that

$$\bar{d}\left(\mathrm{card}(A_1)^{-1} \sum_{j \in A_1} \delta(\sigma^{jr} x_n) , \bar{\mu}_n\right) \leq \varepsilon_n ,$$

and

$$\bar{d}(\ \mathrm{card}(A_2)^{-1}\sum_{j\in A_2}\delta(\sigma^{jr}x_{n+1})\ ,\ \bar{\mu}_{n+1})\leq\varepsilon_{n+1}\ .$$

By the definition of metric $\bar{d}$, we get that

$$\bar{d}(\ \tfrac{1}{k}\sum_{j\in A}\delta(\sigma^{jr}x)\ ,\ \ \mathrm{card}(A_1\cup A_2)^{-1}\sum_{j\in A_1\cup A_2}\delta(\sigma^{jr}x)\ )$$

$$<2\ \mathrm{card}(A_1)^{-1}\{k-\mathrm{card}(A_1\cup A_2)\}\leq\frac{4}{(n+1)!}+2\varepsilon_n\ .$$

Since $d(\sigma^{jr}x,\sigma^{jr}x_n)\leq\varepsilon_n$ $(j\in A_1)$ and $d(\sigma^{jr}x,\sigma^{jr}x_n)\leq\varepsilon_{n+1}$ $(j\in A_2)$, it is easy to check that

$$\bar{d}(\ \tfrac{1}{k}\sum_{j\in A}\delta(\sigma^{jr}x)\ ,\ \ \mathrm{card}(A_1\cup A_2)^{-1}\sum_{j\in A_1}\delta(\sigma^{jr}x_n)$$

$$+\sum_{j\in A_1}\delta(\sigma^{jr}x_{n+1})\ )\ )<\frac{4}{(n+1)!}+2\varepsilon_n$$

$$+\ \bar{d}(\ \mathrm{card}(A_1\cup A_2)^{-1}\sum_{j\in A_1\cup A_2}\delta(\sigma^{jr}x)\ ,\ \ \mathrm{card}(A_1\cup A_2)^{-1}$$

$$(\sum_{j\in A_1}\delta(\sigma^{jr}x_n)+\sum_{j\in A_2}\delta(\sigma^{jr}x_{n+1})\ )\ )$$

$$<\frac{4}{(n+1)!}+3\varepsilon_n+\varepsilon_{n+1}\ .$$

Thus we can compute that

$$\bar{d}(\ \tfrac{1}{k}\sum_{j\in A}\delta(\sigma^{jr}x),\ \mathrm{card}(A_1\cup A_2)^{-1}(\ \mathrm{card}(A_1)\bar{\mu}_n+\mathrm{card}(A_2)\bar{\mu}_{n+1})\ )$$

$$<\frac{4}{(n+1)!}+3\varepsilon_n+\varepsilon_{n+1}+\bar{d}(\ \mathrm{card}(A_1\cup A_2)^{-1}(\sum_{j\in A_1}\delta(\sigma^{jr}x_n)$$

$$+ \sum_{j \in A_2} \delta(\sigma^{jr} x_{n+1})), \ \mathrm{card}(A_1 \cup A_2)^{-1}(\mathrm{card}(A_1)\bar{\mu}_{n+1} + \mathrm{card}(A_2)\bar{\mu}_{n+1})$$

$$\leqslant \frac{4}{(n+1)!} + 4\varepsilon_n + 2\varepsilon_{n+1} .$$

Since $\bar{d}(\bar{\mu}_n, \bar{\mu}_{n+1}) \leqslant \varepsilon_n + \varepsilon_{n+1}$ by (a) , we have that

$$\bar{d}(\tfrac{1}{k}\sum_{j \in A} \delta(\sigma^{jr} x), \bar{\mu}_n) < \frac{4}{(n+1)!} + 5\varepsilon_n + 3\varepsilon_{n+1} .$$

Since $n \geqslant N_0$ and $b_n /r < k \leqslant b_{n+1} /r$ is arbitrary, $V_{\sigma^r}(x)$ coincides the ω-limit set of the sequence $\{\bar{\mu}_n\}_{n=1}$ and so coinsides V by (b). The proof is completed.

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