

On a New Series of Integrable Nonlinear  
Evolution Equations

Yoshi H. Ichikawa\*, Miki Wadati\*\*,  
Kimiaki Konno\*\*\* and Tohru Shimizu\*\*\*\*

- \* Institute of Plasma Physics, Nagoya University, Nagoya  
464.
- \*\* Institute of Physics, College of General Education,  
The University of Tokyo, Tokyo 153.
- \*\*\* Department of Physics, College of Science and Technology,  
Nihon University, Tokyo 101.
- \*\*\*\* Institute of Physics, University of Tsukuba, Ibaraki  
300-31.

Recent results of our research are surveyed in this report. The derivative nonlinear Schrödinger equation for the circular polarized Alfvén wave admits the spiky soliton solutions for the plane wave boundary condition. The nonlinear equation for complex amplitude associated with the carrier wave is shown to be a generalized nonlinear Schrödinger equation, having the ordinary cubic nonlinear term and the derivative of cubic nonlinear term. A generalized scheme of the inverse scattering transformation has confirmed that superposition of the A-K-N-S scheme and the K-N scheme for the component equations valids for the generalized nonlinear Schrödinger equation.

Then, two types of new integrable nonlinear evolution equation have been derived from our scheme of the inverse scattering transformation. One is the type of nonlinear Schrödinger equation, while the other is the type of Korteweg-de Vries equation. Brief discussions are presented for physical phenomena, which could be accounted by the second type of the new integrable nonlinear evolution equation.

Lastly, the stationary solitary wave solutions have been constructed for the integrable nonlinear evolution equation of the second type. These solutions have peculiar structure that they are singular and discreet. It is a new challenge to construct singular potentials by the inverse scattering transformation.

## §1. Introduction

Plasma under the external magnetic fields are unique media that could sustain many kinds of oscillations, of which amplitude attain easily such a large level that nonlinear effects prevail in competition with dispersive effects of the waves. Basing on the reductive perturbation theory, Taniuti and his collaborators<sup>1)</sup> have derived the Korteweg-de Vries equation for the ion acoustic wave, the cubic nonlinear Schrödinger equation for the electron Langmuir wave and the modified Korteweg-de Vries equation for the Alfvén wave. Yet, these do not exhaust all of the possible varieties of nonlinear evolution equations describing the nonlinear wave propagation in plasmas. In fact, the derivative nonlinear Schrödinger equation has been derived for the Alfvén wave propagating parallel to the magnetic field<sup>2)</sup>.

It was the great success of Ablowitz, Kaup, Newell and Segur<sup>3)</sup> that they could find a unified scheme of the inverse scattering transformation for the above mentioned canonical type of nonlinear evolution equations. Since the derivative nonlinear Schrödinger equation is unable to be casted into the A-K-N-S scheme, we have carried out detailed analysis of the stationary solution of this new type of nonlinear evolution equation<sup>4),5)</sup>. Then, Kaup and Newell<sup>6)</sup> have presented a new scheme of the inverse scattering transformation for the derivative nonlinear Schrödinger equation.

According to our analysis<sup>5)</sup>, however, the solutions we have obtained are the stationary solutions of a generalized nonlinear Schrödinger equation, of which nonlinear terms are sum of the cubic nonlinear term and the derivative nonlinear term. Through a generalization of the inverse scattering transformation, we have shown explicitly that the inverse scattering transformation for the generalized nonlinear Schrödinger equation is nothing but a linear superposition of the A-K-N-S scheme and K-N scheme<sup>7)</sup>.

Furthermore, within the scheme of our generalization of the inverse scattering transformation, we have discovered two types of integrable nonlinear evolution equations, both of which have saturative nonlinear terms<sup>8)</sup>.

In the present report, we summarize recent development of our studies, and will discuss the novel feature of soliton solutions of these new integrable nonlinear evolution equations.

## §2. Spiky Solitary Waves of Alfvén Wave

The circularly polarized Alfvén wave propagating along the magnetic field obeys the derivative nonlinear Schrödinger equation<sup>2)</sup>

$$i \frac{\partial}{\partial t} q + \mu \frac{\partial^2}{\partial \xi^2} q + i \frac{1}{4} \frac{\partial}{\partial \xi} \{|q|^2 q\} = 0, \quad 1)$$

where  $q$  is the complex magnetic field

$$q = B_y \mp iB_z \quad , \quad 2)$$

and  $\xi$  is the moving coordinate with the Alfvén velocity in  $x$ -direction.

We seek stationary solutions of eq.1) describing the nonlinear self-modulation of large amplitude plane wave<sup>5)</sup>.

Setting  $q$  as

$$q(\xi, t) = \sqrt{8}\psi(\xi, t)\exp[i\chi(\xi, t)] \quad , \quad 3.a)$$

$$\chi(\xi, t) = (k\xi - \Omega t) + \theta(y) \quad , \quad 3.b)$$

$$\psi(\xi, t) = \psi(y) \quad , \quad 3.c)$$

with

$$y = \xi - \lambda t \quad , \quad 3.d)$$

we obtain

$$\psi^2(y) \equiv \phi(y) = \phi_0 + 8\kappa\gamma^2\beta^{-1} \{ \kappa m + \cosh[2\gamma|\mu|^{-1/2}(y-y_0)] \}^{-1} \quad , \quad 4.a)$$

$$\theta(y) = \theta(y_0) + 3\kappa \frac{|\mu|^{1/2}}{\mu} \arctan \left\{ \sqrt{\frac{1-\kappa m}{1+\kappa m}} \tanh[\gamma|\mu|^{-1/2}(y-y_0)] \right\} \\ + \kappa\delta \frac{|\mu|^{1/2}}{\mu} \arctan \left\{ \sqrt{\frac{1-\kappa\ell}{1+\kappa\ell}} \tanh[\gamma|\mu|^{-1/2}(y-y_0)] \right\} \quad , \quad 4.b)$$

where

$$\kappa = \pm 1, \quad 5.a)$$

$$\delta = \text{sign of } (3\phi_0 - \lambda - 2\mu\kappa) \quad , \quad 5.b)$$

$$\alpha = 2\{2\phi_0 - \lambda - 2(1+\mu)\kappa\} \quad , \quad 5.c)$$

$$\beta = 4\{(\phi_0 + \kappa)[\lambda + (1+2\mu)\kappa - 2\phi_0]\}^{1/2} \quad , \quad 5.d)$$

$$\gamma^2 = \frac{1}{4} (\lambda - \lambda_1)(\lambda_2 - \lambda) \quad , \quad 5.e)$$

$$l = \alpha/\beta + 8\gamma^2/(\beta\phi_0) \quad , \quad 5.f)$$

$$m = \alpha/\beta \quad .$$

The allowed range of propagation velocity  $\lambda$  is determined to be

$$\lambda_1 < \lambda < \lambda_2 \quad 6)$$

with the definition of

$$\lambda_1 = 2(2\phi_0 - \mu k) - 2\sqrt{\phi_0(\phi_0 + k)} \quad 7.a)$$

$$\lambda_2 = 2(2\phi_0 - \mu k) + 2\sqrt{\phi_0(\phi_0 + k)} \quad 7.b)$$

Referring to the left polarized Alfvén wave, we illustrate characteristic feature of the solitary wave solutions given by eqs.4.a) and 4.b). For arbitrary chosen parameters of  $\kappa=+1$ ,  $\sqrt{\phi_0}=0.5/\sqrt{8}$ ,  $k=0.01$  and  $\mu=0.5$ ,  $\lambda=2(2\phi_0-\mu k)$ , Fig.1 represents the bright hyperbolic solitary wave, resulting from the self-modulation of

---

Fig.1

---

the large amplitude plane wave, due to a strong coupling of nonlinear modulation of the amplitude and the phase. When the propagation velocity  $\lambda$  takes the limiting velocities  $\lambda_1$  or  $\lambda_2$ , eqs.(7.a), (7.b), the hyperbolic solitary wave solution is reduced to the algebraic solitary wave solution,

$$\psi^2(y) \equiv \phi(y) = \phi_0 + \frac{4\rho}{4 + \rho^2 |\mu|^{-1} (y - y_0)^2} \quad 8.a)$$

$$\theta(y) = \theta(y_0) + \frac{|\mu|^{1/2}}{\mu} \varepsilon \arctan\left(\frac{1}{2}\rho v |\mu|^{-1/2} (y - y_0)\right) + \frac{|\mu|^{1/2}}{\mu} 3 \arctan\left(\frac{1}{2}\rho |\mu|^{-1/2} (y - y_0)\right) \quad 8.b)$$

where

$$\rho = 4(\phi_0 + k) + \varepsilon 4\sqrt{\phi_0(\phi_0 + k)} \quad 9.a)$$

$$v = \sqrt{\phi_0} / |2\sqrt{\phi_0 + k} + \varepsilon\sqrt{\phi_0}| \quad 9.b)$$

$$\varepsilon = \begin{cases} +1 & \text{for } \lambda = \lambda_2 \\ -1 & \text{for } \lambda = \lambda_1 \end{cases} \quad 9.c)$$

Fig. 2 illustrates the fast algebraic solitary wave moving with the velocity  $\lambda_2$  for the same values of  $\psi_0$ ,  $k$  and  $\mu$  as in Fig. 1.

---

Fig. 2

---

In order to show physical differences between the derivative nonlinear Schrödinger equation for the Alfvén wave and the cubic nonlinear Schrödinger equation, we substitute

$$q(\xi, t) = Q(\eta, t) \exp[i(k\eta - \mu k^2 t)], \quad 10.a)$$

with

$$\eta = \xi - 2\mu k t, \quad 10.b)$$

into eq. (1). Then, we get an equation for the complex

amplitude  $Q(\eta, t)$  as

$$i \frac{\partial Q}{\partial t} + \mu \frac{\partial^2 Q}{\partial \eta^2} - \frac{k}{4} |Q|^2 Q + \frac{i}{4} \frac{\partial}{\partial \eta} \{ |Q|^2 Q \} = 0, \quad (11)$$

where the nonlinear terms are composed of the usual cubic nonlinear term and the derivative nonlinear term. The spiky solitary waves presented as eqs. 4.a), 4.b) and 8.a), 8.b) are the stationary state solution of eq. 11).

### §3. A Generalization of Inverse Scattering Method

In spite of the success of Ablowitz, Kaup, Newell and Segur<sup>3)</sup> to present a unified scheme of the inverse scattering transformation for a certain class of nonlinear evolution equations, their scheme appears to be not general enough to cover large varieties of the integrable nonlinear evolution equation. Indeed, Kaup and Newell<sup>6)</sup> have presented another scheme for the derivative nonlinear Schrödinger equation and the massive Thirring equation. Inspired by the solitary wave solutions of the generalized nonlinear Schrödinger equation, eq. (11), we have undertaken to seek a generalization of the inverse scattering transformation to cover a wider class of nonlinear evolution equations<sup>7)</sup>.

We consider the eigenvalue problem.

$$\frac{\partial}{\partial x} v_1 + F(\lambda) v_1 = G(\lambda) q(x, t) v_2 \quad 12.a)$$



$$\frac{\partial}{\partial x} v_2 - F(\lambda) v_2 = G(\lambda) r(x, t) v_1 \quad 12.b)$$

where  $F(\lambda)$  and  $G(\lambda)$  are functions of the eigenvalue  $\lambda$ . The A-K-N-S scheme is a special case of our generalization,  $F(\lambda) = i\lambda$  and  $G(\lambda) = 1$ . The time dependence of the eigenfunctions is chosen to be

$$\frac{\partial}{\partial t} v_1 = A(\lambda, q, r) v_1 + B(\lambda, q, r) v_2, \quad 13.a)$$

$$\frac{\partial}{\partial t} v_2 = C(\lambda, q, r) v_1 - A(\lambda, q, r) v_2. \quad 13.b)$$

Noting that  $(v_{ix})_t = (v_{it})_x$ ,  $i=1,2$ , and assuming that the eigenvalues  $\lambda$  are time invariant we readily find that  $A(\lambda, q, r)$ ,  $B(\lambda, q, r)$  and  $C(\lambda, q, r)$  satisfy

$$A_x + G(rB - qC) = 0 \quad 14.a)$$

$$Gq_t - B_x - 2FB - 2GqA = 0 \quad 14.b)$$

$$Gr_t - C_x + 2FC + 2GrA = 0 \quad 14.c)$$

The proper choices of  $A$ ,  $B$ ,  $C$ ,  $F$  and  $G$  yield various integrable nonlinear evolution equations.

Referring to the A-K-N-S scheme and the K-N scheme, we choose

$$F(\lambda) = i\alpha\lambda^2 - \sqrt{2\beta} \lambda, \quad 15.a)$$

$$G(\lambda) = \alpha\lambda + i\sqrt{\beta/2}, \quad 15.b)$$

Namely we consider the superposition of these two scheme of the inverse scattering transformation. Here,  $\alpha$  and  $\beta$  are positive constants. Then, with the choice of

$$A(\lambda, q, r) = -2i\alpha^2 \lambda^4 + 4\alpha\sqrt{2\beta} \lambda^3 + (4i\beta - i\alpha^2 r q) \lambda^2 + \sqrt{2\beta} \alpha r q \lambda + i(\beta/2) r q, \quad 16.a)$$

$$B(\lambda, q, r) = 2\alpha^2 q \lambda^3 + 3i\sqrt{2\beta} \alpha q \lambda^2 + (-2\beta q + i\alpha q_x + \alpha^2 r q^2) \lambda + (-\sqrt{\beta/2} q_x + i\alpha\sqrt{\beta/2} r q^2), \quad 16.b)$$

$$C(\lambda, q, r) = 2\alpha^2 r \lambda^3 + 3i\sqrt{2\beta} \alpha r \lambda^2 + (-2\beta r - i\alpha r_x + \alpha^2 r^2 q) \lambda + (\sqrt{\beta/2} r_x + i\alpha\sqrt{\beta/2} r^2 q), \quad 16.c)$$

We obtain from eqs. (14.b) and (14.c) the set of nonlinear evolution equations

$$iq_t + q_{xx} - i\alpha(rq^2)_x + \beta r q^2 = 0, \quad 17.a)$$

$$ir_t - r_{xx} - i\alpha(r^2 q)_x - \beta r^2 q = 0. \quad 17.b)$$

If we take  $r = \underline{q}^*$ , the set of equations (17.a) and (17.b) is reduced to the generalized nonlinear Schrödinger equation

$$iq_t + q_{xx} = i\alpha(|q|^2 q)_x + \beta|q|^2 q = 0. \quad 18)$$

The above analysis is the first illustration of the fact that the linear superposition of two different schemes of the inverse scattering transformation works out to solve the nonlinear evolution equation with mixed type of nonlinear terms.

#### 54. New Integrable Nonlinear Evolution Equations

Having been encouraged by the success of our generalization of the inverse scattering transformation, we have undertaken

to explore new types of integrable nonlinear evolution equations<sup>8)</sup>.

We assign for  $F(\lambda)$  and  $G(\lambda)$  of eqs.(12.a) and (12.b) the following expressions,

$$F(\lambda) = i\lambda \quad , \quad 19.a)$$

$$G(\lambda) = \lambda \quad . \quad 19.b)$$

Then, choosing

$$A = - \frac{2i}{\sqrt{1-rq}} \lambda^2 \quad , \quad 20.a)$$

$$B = 2 \frac{q}{\sqrt{1-rq}} \lambda^2 + i \left( \frac{q}{\sqrt{1-rq}} \right)_{xx} \lambda \quad , \quad 20.b)$$

$$C = 2 \frac{r}{\sqrt{1-rq}} \lambda^2 - i \left( \frac{r}{\sqrt{1-rq}} \right)_{xx} \lambda \quad , \quad 20.c)$$

We obtain from eqs.(14.b) and (14.c)

$$q_t - i \left( \frac{q}{\sqrt{1-rq}} \right)_{xx} = 0 \quad 21.a)$$

$$r_t + i \left( \frac{r}{\sqrt{1-rq}} \right)_{xx} = 0 \quad 21.b)$$

If we take  $r=+q^*$ , eqs.(21.a) and (21.b) are reduced to

$$iq_t + \frac{\partial^2}{\partial x^2} \left\{ \frac{q}{\sqrt{1+|q|^2}} \right\} = 0 \quad . \quad 22)$$

Another choices of A, B and C,

$$A = - \frac{4i}{\sqrt{1-rq}} \lambda^3 + \frac{rq_x - qr_x}{3/2(1-rq)} \lambda^2 \quad , \quad 23.a)$$

$$B = \frac{4q}{\sqrt{1-rq}} \lambda^3 + \frac{2iq_x}{(1-rq)^{3/2}} \lambda^2 - \left( \frac{q_x}{(1-rq)^{3/2}} \right) x \lambda, \quad 23.b)$$

$$C = \frac{4r}{\sqrt{1-rq}} \lambda^3 - \frac{2ir_x}{(1-rq)^{3/2}} \lambda^2 - \left( \frac{r_x}{(1-rq)^{3/2}} \right) x \lambda, \quad 23.c)$$

lead to the following set of equations

$$q_t + \left( \frac{\partial^2}{\partial x^2} \frac{1}{(1-rq)^{3/2}} \frac{\partial q}{\partial x} \right) = 0, \quad 24.a)$$

$$r_t + \left( \frac{\partial^2}{\partial x^2} \frac{1}{(1-rq)^{3/2}} \frac{\partial r}{\partial x} \right) = 0. \quad 24.b)$$

If we take  $r=-q$ , the set of eqs.(24.a) and (24.b) is reduced to

$$\frac{\partial}{\partial t} q + \frac{\partial^2}{\partial x^2} \left( \frac{1}{(1+q)^{3/2}} \frac{\partial q}{\partial x} \right) = 0. \quad 25)$$

We notice also, for  $r=-1$  and  $q=u-1$ , eqs.(24.a) and (24.b) are reduced to

$$\frac{\partial}{\partial t} u = 2 \frac{\partial^3}{\partial x^3} u^{-1/2}, \quad 26)$$

which is the equation discovered by Harry Dym<sup>9)</sup>.

Shimizu and Wadati<sup>10)</sup> have carried out detailed analysis of eq.(22) with the lower sign under the boundary condition

$$q(x,t) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty. \quad 27)$$

For the bound state eigenvalue  $\lambda_1$  given as

$$\lambda_1 = \xi + i\eta, \quad \eta > 0, \quad (28)$$

an expression for one-soliton solution is obtained;

$$q(x,t) = -\frac{2\eta}{\sqrt{\xi^2 + \eta^2}} \frac{\cosh[2\eta y + 2\eta \varepsilon_+(y) - i\alpha]}{\cosh^2[2\eta y + 2\eta \varepsilon_+(y)] - 2\frac{\eta^2}{\xi^2 + \eta^2}} \\ \times \exp[-2i\{\xi x + 2(\xi^2 - \eta^2)t + \xi \varepsilon_+(y)\}], \quad (29.a)$$

where  $\varepsilon_+(y)$  is given implicitly as

$$y = -\varepsilon_+(y) + \frac{1}{4\eta} \log \left[ -\frac{2\eta}{\xi^2 + \eta^2} \frac{1}{\varepsilon_+(y)} - 1 \right]. \quad (29.b)$$

The quantity  $y$  is a moving coordinate defined by

$$y = x + 4\xi t. \quad (30)$$

As for the illustration, the function  $\varepsilon_+(y)$  is shown in Fig. 3 for  $\eta=1/2$  and  $\xi/\eta=\sqrt{3}$ .

---

Fig.3

---

Since  $\varepsilon_+(y)$  is determined numerically from eq.(29.b), we can evaluate  $q(x,t)$  numerically. Figure 4 illustrates the one soliton envelope  $|q(x,t)|$  for  $\eta=1/2$  with various values

---

Fig.4

---

of  $\xi/\eta$ . For the small amplitude limit of  $\eta \ll \xi$ , eqs.(29.a) and (29.b) are reduced to

$$q(x,t) = \frac{2\eta}{\xi} \operatorname{sech}[2\eta(x+4\xi t)] \exp[-2i\xi x - 4i(\xi^2 - \eta^2)t], \quad 31)$$

which is the one soliton solution of the cubic nonlinear Schrödinger equation

$$iq_t + q_{xx} + 2\xi^2 |q|^2 q = 0. \quad 32)$$

In Fig.4, the limit of a bursting soliton given as  $|\xi| \rightarrow \eta + 0$  has been shown, as well.

#### §5. Some Physical Problems Related to Eq.(25)

Although the new integrable nonlinear evolution equations bear novel feature for the mathematical interest, we have tried to identify physical problems for which the new integrable evolution equations are relevant. We find the eq.(25) could be the key equation for physical problems in which the curvature of surface or any deformation is crucial.

One of such problems is the nonlinear transverse oscillation of elastic beams under tension<sup>11)</sup>. We can write down the equations of motion of the small

---

Fig.5

---

element AB illustrated in Fig.5 as

$$\rho A \frac{\partial^2}{\partial t^2} y = \frac{\partial}{\partial x} S, \quad 33.a)$$

$$0 = \frac{\partial}{\partial x} M + P \frac{\partial}{\partial x} y + S ; \quad 33.b)$$

where  $\rho$  is density of material,  $A$  stands for area of cross section.  $S$  is the stress resultant parallel to the axis of  $y$ , and  $P$  is the end-thrust parallel to the axis of  $x$ . As far as we are considering a uniform elastic beam,  $P$  is assumed to be constant.

For the bending moment  $M$ , we have the relation

$$M = \frac{EI}{R} = EI \frac{1}{\{1 + (\partial y / \partial x)^2\}^{3/2}} \frac{\partial^2 y}{\partial x^2}, \quad 34)$$

where  $E$  is Young's modulus,  $R$  represents the radius of curvature of bending beam, and  $I$  is the moment of inertia of the cross section of beam. When the beam subjects to tension  $P = -\mathcal{J}$ , combining eqs. (33.a), (33.b) and (34), we obtain the following nonlinear partial differential equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\mathcal{J}}{\rho A} \frac{\partial^2 y}{\partial x^2} + \frac{EI}{\rho A} \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial^2 y / \partial x^2}{[1 + (\partial y / \partial x)^2]^{3/2}} \right\} = 0. \quad 35)$$

Since we are dealing with the uniform elastic beam, we do not have any dynamical nonlinear effect, but we have taken fully account of the geometrical nonlinear effect.

Introducing the dimensionless variables  $X$ ,  $Y$  and  $T$  as

$$x = A^{1/2} X \quad 36.a)$$

$$y = A^{1/2} Y \quad 36.b)$$

$$t = (A^{1/2} / \lambda) T, \quad 36.c)$$

where  $\lambda = (\mathcal{J} / \rho A)^{1/2}$  is the linear wave velocity, and defining the stretched coordinates

$$\xi = X + T \quad 37.a)$$

$$\tau = \varepsilon T \quad 37.b)$$

with the dimensionless parameter  $\varepsilon$  given as

$$\varepsilon = \frac{EI}{2\gamma A}, \quad 38)$$

we can reduce eq. (35) to, up to the first order of  $\varepsilon$ ,

$$\frac{\partial}{\partial \tau} \frac{\partial}{\partial \xi} Y + \frac{\partial^2}{\partial \xi^2} \left\{ \frac{\partial^2 Y / \partial \xi^2}{[1 + (\partial Y / \partial \xi)^2]^{3/2}} \right\} = 0. \quad 39)$$

Defining

$$q(\xi, \tau) = \frac{\partial Y}{\partial \xi}, \quad 40)$$

we can see immediately that eq. (39) is nothing but the equation of the second types of our new integrable nonlinear evolution equations, that is

$$\frac{\partial}{\partial \tau} q + \frac{\partial^2}{\partial \xi^2} \left\{ \frac{\partial q / \partial \xi}{(1+q^2)^{3/2}} \right\} = 0. \quad 41)$$

Another example could be found in the surface phenomena in which the surface tension plays the key role. For instance<sup>12</sup> the shape of the surface of a fluid in a gravitational field and bounded on one side by a vertical plane wall is determined by

$$\frac{2}{a} z - \frac{z''}{(1+z'^2)^{3/2}} = \text{const}, \quad 42)$$

where

$$a = \sqrt{(2\alpha/\rho g)} \quad 43)$$

is called the capillary constant.  $\alpha$  is the surface tension coefficient, and  $g$  is the gravitational acceleration.



Defining the function  $q(x)$  by

$$q = \frac{\partial z}{\partial x}, \quad (44)$$

and differentiating eq.(42) twice with respect to  $x$ , we obtain

$$\frac{2}{a^2} \frac{\partial}{\partial x} q - \frac{\partial^2}{\partial x^2} \left\{ \frac{q'}{(1+q^2)^{3/2}} \right\} = 0. \quad (45)$$

Therefore, eq.(43) could be regarded as the stationary solution of eq.(41), which depends on the single variable

$$\eta = \xi - \frac{2}{a^2} \tau. \quad (46)$$

Of course, in order to apply the inverse scattering method to integrate eqs.(41) or (45), we have to carefully consider the boundary conditions of the problems.

#### §6. Analysis of Stationary Solutions of Eq.(41)

Here, let us examine stationary solutions of eq.(41) assuming that  $q$  depends on a single parameter

$$\eta = \xi \pm v\tau, \quad v > 0. \quad (47)$$

Substituting the transformation (47) into eq.(41), we can carry out the first integration to get

$$vq \pm \frac{d}{d\eta} \left\{ (1+q^2)^{-3/2} \frac{d}{d\eta} q \right\} = 0 \quad (48)$$

for the boundary conditions of  $q=0$  and  $dq/d\eta=0$  at  $|\eta| \rightarrow \infty$ .

Setting

$$q = \frac{dy}{d\eta}, \quad (49)$$

we can again integrate eq.(48) to yield

$$vY \pm \left\{1 + \left(\frac{dY}{d\eta}\right)^2\right\}^{-3/2} \frac{d^2}{d\eta^2} Y = 0. \quad (50)$$

Multiplying  $dY/d\eta$  on the both side of eq.(50), we can get

$$\frac{v}{2} Y^2 \mp \left\{1 + \left(\frac{dY}{d\eta}\right)^2\right\}^{-1/2} = C. \quad (51)$$

For the lower sign of eq.(51), we can obtain a localized solution by imposing the boundary conditions at the infinity as

$$\frac{dY}{d\eta} = 0, \quad Y = 0 \quad \text{at} \quad \eta \rightarrow \pm\infty. \quad (52)$$

This boundary condition specifies the value of C as

$$C = 1. \quad (53)$$

We assign the following boundary conditions at a point  $\eta_0$ ,

$$\frac{dY}{d\eta} = -\cot\theta, \quad Y = Y_0 \quad \text{at} \quad \eta \rightarrow \eta_0 + 0, \quad (54.a)$$

$$\frac{dY}{d\eta} = +\cot\theta, \quad Y = Y_0 \quad \text{at} \quad \eta \rightarrow \eta_0 - 0, \quad (54.b)$$

where  $\theta$  is an arbitrary constant. For a given value of  $\theta$ ,  $Y_0$  is determined to be

$$Y_0 = \pm \left(\frac{2}{v}\right)^{1/2} (1 - \sin\theta)^{1/2}. \quad (55)$$

For the positive sign of eq.(55), carrying out the integration of  $dY/d\eta$  derived from eq.(51), we obtain

$$\begin{aligned} \pm\sqrt{v}(\eta - \eta_0) = & -\operatorname{sech}^{-1} \left| \frac{\sqrt{v}}{2} Y \right| + 2 \left(1 - \frac{v}{4} Y^2\right)^{1/2} \\ & + \operatorname{sech}^{-1} \left| \frac{\sqrt{v}}{2} Y_0 \right| - 2 \left(1 - \frac{v}{4} Y_0^2\right)^{1/2}. \end{aligned} \quad (56)$$

In Fig.6, we illustrate the solution  $Y(\eta)$

---

Fig.6

---

for  $\theta=0$ ,  $\theta=\pi/4$  and  $\theta=\pi/3$ . The solitary wave solution of eq. (41),  $q(\eta)$ , is calculated from eq.(49) with eq.(51).

Corresponding to the

---

Fig.7

---

three cases of Fig.6, we illustrate in Fig.7 the corresponding solutions. These are singular and discrete solitary waves. Although such singular solutions are not acceptable as physical solutions for the nonlinear transverse oscillation of the elastic beam, the present analysis poses the challenging problem whether we can construct such solutions by the inverse scattering transformation.

## §7. Concluding Discussions

In the present report, we have surveyed recent results of our investigation on the nonlinear evolution equations. Generalization of the inverse scattering transformation has demonstrated that the linear superposition of two schemes of the inverse scattering transformation works to solve the

generalized nonlinear evolution equation with the superposed nonlinear terms, each of which corresponds the component scheme of the inverse scattering transformation. We need to explore to what extent such superposition works out to solve the nonlinear evolution equation.

In this connection, it may be worth to mention the work of Hirota<sup>13)</sup>. He had shown that the generalized nonlinear equation

$$iq_t + \rho q_{xx} + i\sigma q_{xxx} + \delta |q|^2 q + i3\alpha |q|^2 q_x = 0 \quad (57)$$

has N-soliton solutions, provided that the condition

$$\alpha\rho = \sigma\delta \quad (58)$$

holds. Eq.(57) is a superposition of the nonlinear Schrödinger equation and the complex modified Korteweg-de Vries equation. For the special case of  $\rho=0$  and  $\delta=0$ , eq.(57) is known to have multiple envelope soliton solutions<sup>14)</sup>. On the other hand, Chen, Lee and Liu<sup>15)</sup> have shown that eq.(57) is integrable for the special case of  $\sigma=0$  and  $\delta=0$ , which violates the condition eq.(58).

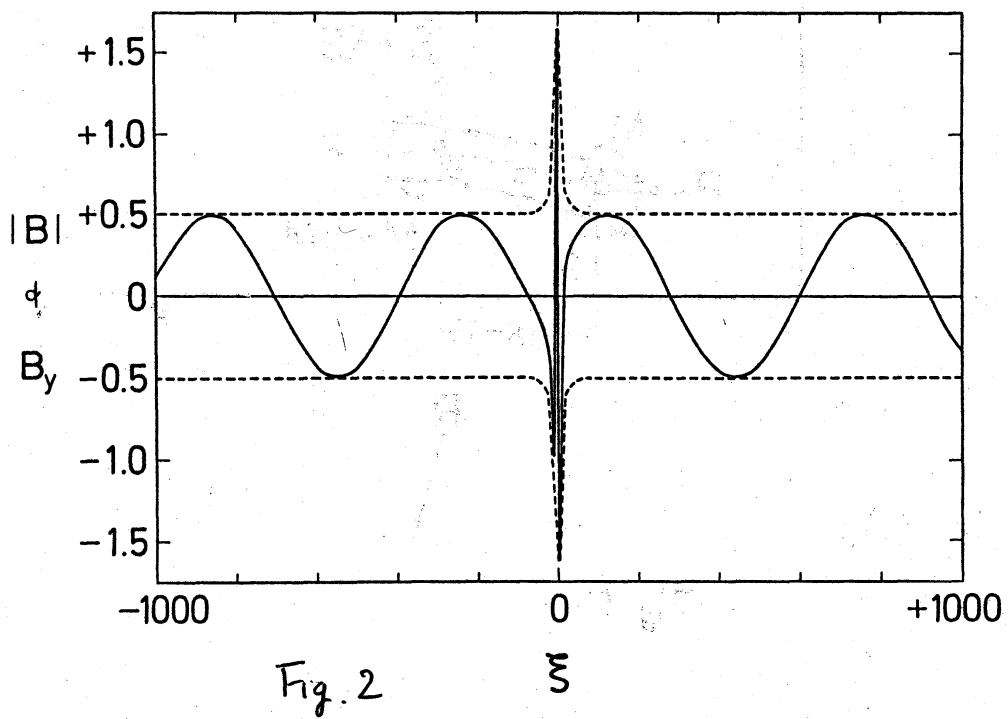
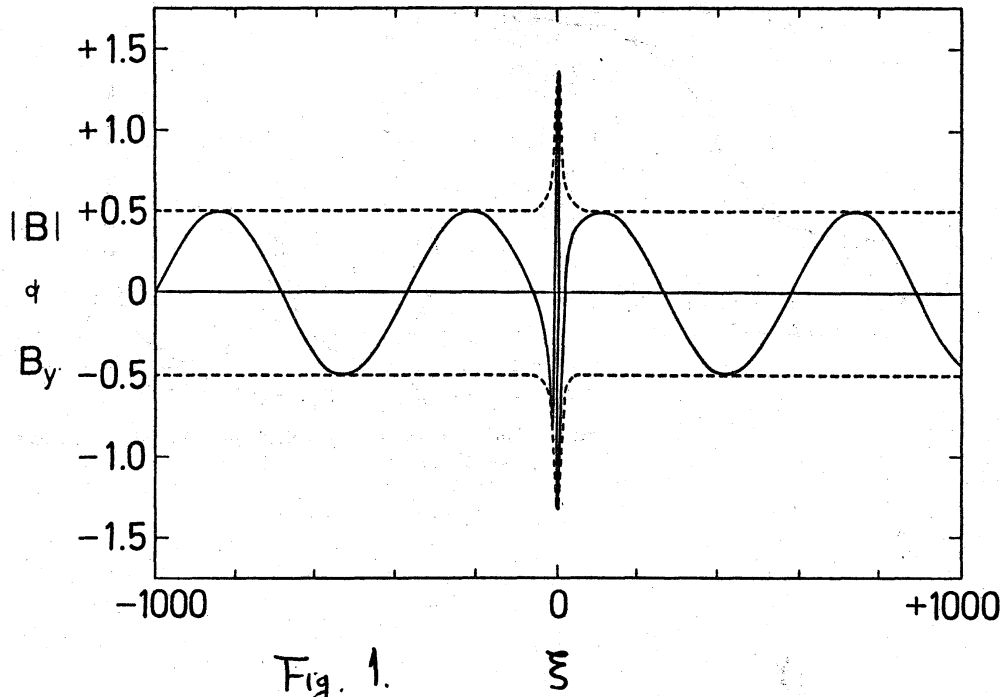
Studies of the new integrable nonlinear evolution equation have revealed the new problem of the inverse scattering transformation to construct singular potentials. If the singular solitary waves and the discrete solitary waves demonstrated in the section 6 could obtain the citizenship of the soliton empire, the extension of soliton science will be far more reaching than that was expected when Zabusky and Kruskal<sup>16)</sup> had coined the concept of soliton for the solitary wave of the Korteweg-de Vries equation.

## References

1. T. Taniuti (editor): Suppl. Prog. Theor. Phys. 55 (1974)
2. A. Rogister: Phys. Fluids 14 (1971) 2733
3. M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: Stud. Appl. Math. 53 (1974) 249
4. M. Wadati, H. Sanuki, K. Konno and Y. H. Ichikawa: Proc. of the Conf. on Theory and Application of Solitons (Tucson, Arizona, Jan 5-10, 1976). Rocky Mountain Math. Journal 8 (1978) 323
5. Y. H. Ichikawa, K. Konno, M. Wadati and H. Sanuki: J. Phys. Soc. Japan 48 (1980) 279
6. D. J. Kaup and A. C. Newell: J. Math. Phys. 19 (1978) 798
7. M. Wadati, K. Konno and Y. H. Ichikawa: J. Phys. Soc. Japan 46 (1979) 1965
8. M. Wadati, K. Konno and Y. H. Ichikawa: J. Phys. Soc. Japan 47 (1979) 1698
9. M. D. Kruskal: Lecture Notes in Physics 38 (1975) 310
10. T. Shimizu and M. Wadati: Prog. Theor. Phys. 63 (1980) 808
11. Y. H. Ichikawa, K. Konno and M. Wadati: preprint
12. L. D. Landau and E. M. Lifshitz: Fluid Mechanics (Pergamon Press, 1959) Chap. 7
13. R. Hirota: J. Math. Phys. 14 (1973) 805
14. R. Hirota: J. Math. Phys. 14 (1973) 810
15. H. H. Chen, Y. C. Lee and C. S. Liu: Physica Scripta 20 (1979) 490
16. N. J. Zabusky and M. D. Kruskal: Phys. Rev. Lett. 15 (1965) 240

## Captions of Figures

- Fig.1 The bright hyperbolic solitary wave of the left polarized Alfvén wave for a value of  $\lambda=2(2\phi_0-\mu k)$  with an arbitrary chosen phase constant  $\theta(y_0)=\pi/6$ . The thin line represents the magnetic field component  $B_y$  and the dotted line stands for  $|B|$ , respectively.
- Fig.2 The fast algebraic solitary wave of the left polarized Alfvén wave with a propagation velocity  $\lambda_2$ .
- Fig.3 The curve of  $\varepsilon_+(u)$  for  $\eta=1/2$  and  $\xi/\eta=\sqrt{3}$ .
- Fig.4 The envelope of one-soliton solution for  $\eta=1/2$  and  $\xi/\eta=\sqrt{3}, \sqrt{2}$  and 1.
- Fig.5 Transverse displacement of elastic beam under the end-thrust.
- Fig.6 Localized stationary solutions of eq.(39) for  $\theta=0$ ,  $\theta=\pi/4$  and  $\theta=\pi/3$ .
- Fig.7 Singular solitary wave solutions of eq.(41) derived from the solution Y with  $\theta=0$ ,  $\theta=\pi/4$  and  $\theta=\pi/3$ , respectively.



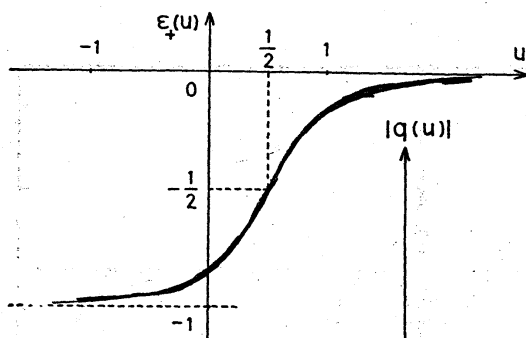


Fig. 3 The curve of  $\epsilon_+(u)$  for  $\eta=1/2$  and  $\xi/\eta=\sqrt{3}$ .

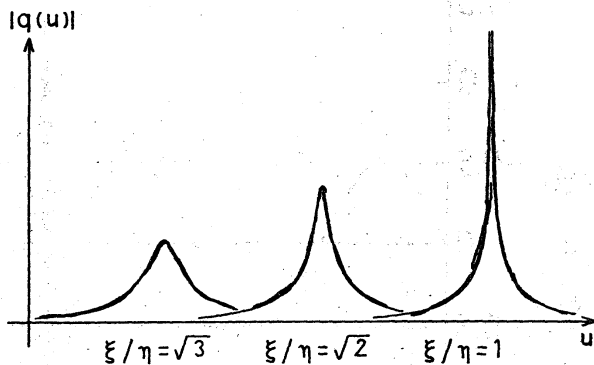


Fig. 4 The envelope of one-soliton solution for  $\eta=1/2$  and  $\xi/\eta=\sqrt{3}, \sqrt{2}$  and 1.

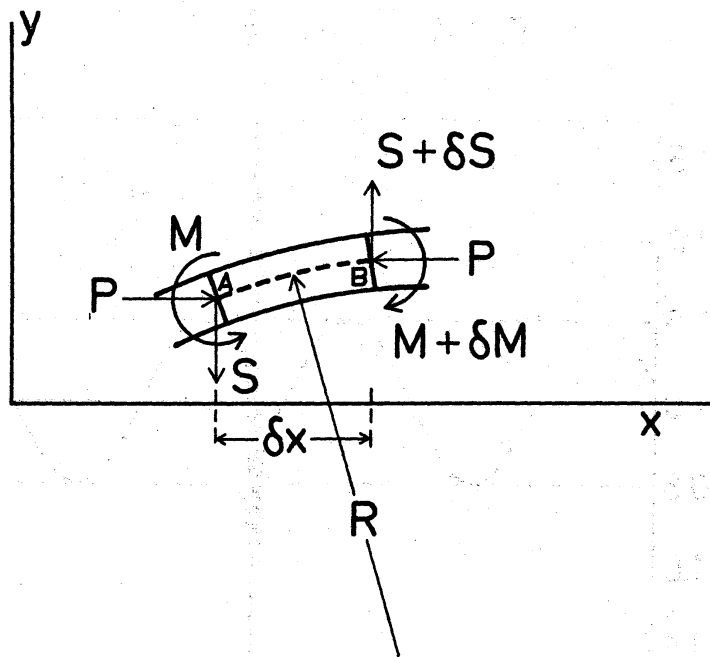


Fig. 5



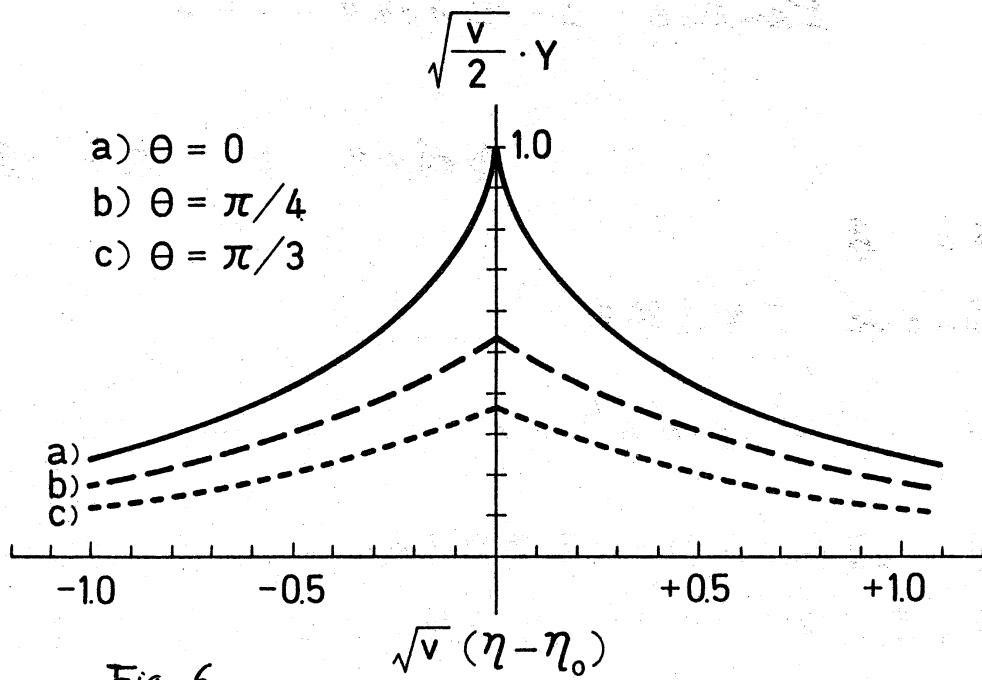


Fig. 6.

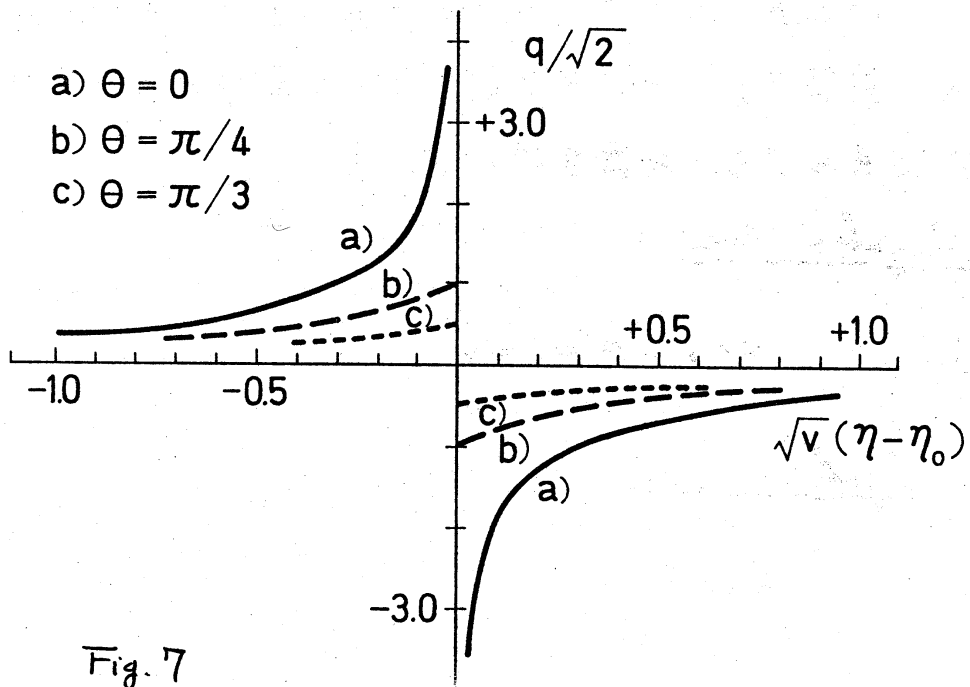


Fig. 7