数理解析研究所講究録
第 414 巻 1981 年 171-180



# Composite Structures of a Class of Soliton Solutions

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## §1. Introduction

We have been investigating our proposition that nonlinear evolution equations which exhibit solitons are decomposed into the fundamental equations, the bilinear equations. Until now several types of the bilinear equations have been found.

The first type involves only one dependent variable f and is expressed with the binary operaters

$$F(D_x, D_t, \ldots)\, f\cdot f = 0, \tag{1.1}$$

where F is a polynomial or exponential function of $D_x$, $D_t$, etc., and $D_x$, $D_t$ are the binary operators defined by

$$D_x^m D_t^n f(x,t)\cdot g(x,t) \equiv \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,t)\, g(x',t')\Big|_{\substack{x'=x\\ t'=t}}$$

The nonlinear differential equations which are transformed into eq.(1.1) are

(i) KdV equation.

(ii) Boussinesq equation.

(iii) Kadomtsev-Petviashvili equation.

(iv) Model equations for shallow water waves.

(v) Higher order KdV equations.

(vi) Toda equation.

(vii) Discrete analogue of the KdV equation.

(viii) Discrete-time Toda equation.

The second type involves two dependent variables f' and f, and is expressed as

$$F_i(D_x, D_t, \ldots) f' \cdot f = 0 \quad \text{for } i = 1, 2. \tag{1.3}$$

The nonlinear evolution equations transformed into eq.(1.3) are

(i) Modified KdV equations.

(ii) Sine-Gordon equation.

(iii) Modified Boussinesq equation.

(iv) Benjamin-Ono equation.

(v) Self-dual nonlinear network equation.

(vi) Equations describing a Volterra system.

(vii) Discrete-time nonlinear network equation.

In the present paper we shall discuss on the third type which involvs four dependent variables f', f, g', and g

$$F_{1i}(D_x, D_t) f' \cdot f + F_{2i}(D_x, D_t) g' \cdot g = 0, \tag{1.4a}$$
$$F_{3i}(D_x, D_t) f' \cdot g + F_{4i}(D_x, D_t) f \cdot g' = 0, \tag{1.4b}$$

for i=1,2.

§2. Nonlinear Schrödinger Equation and the Classical Heisenberg Ferromagnet

Let us start with the relation between the nonlinear Schrödinger equation and the one dimensional classical Hiesenberg ferromagnet (Lakshmanan, Takhtjan, 1977). We have the nonlinear Schrödinger equation and the Heisenberg ferromagnet equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{2.1}$$

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}, \quad \|\vec{S}\| = 1, \tag{2.2}$$

where the subscripts indicate partial differnciations with respect to indicated variables.

According to the classical differential geometry, the spatial variation of a twisted curve are governed by the Serret-Frenet equations

$$\frac{d}{dx}\begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix} \tag{2.3}$$

where $x$ is the arc length parameter. The function's $\tau(x, t)$ and $\kappa(x, t)$ are the torsion and curvature, respectively while $\hat{t}$, $\hat{n}$ and $\hat{b}$ are the usual tangent, normal, and binormal to the curve.

Hashimoto (1972), Lamb (1976) found that if the motion of the twisted curve is described by

$$\frac{d}{dt}\begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0 & -\kappa\tau & \kappa_x \\ \kappa\tau & 0 & \frac{\kappa_{xx}}{\kappa} - \tau^2 \\ -\kappa_x & -\frac{\kappa_{xx}}{\kappa} + \tau^2 & 0 \end{bmatrix}\begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix} \tag{2.4}$$

then the comptability condition on eqs.(2.3) and (2.4) gives the evolution equations of $\kappa$ and $\tau$ which are combined to give the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{2.5}$$

where

$$\psi = \kappa \exp[i\int_{-\infty}^{x} \tau(t,x')dx']. \tag{2.6}$$

On the other hand, eleminating $\kappa$ and $\tau$ out of eqs.(2.3) and (2.4) and identifying the tangent vector $\hat{t}$ with the unit spin vector $\vec{S}$, one obtains the Heisenberg ferromagnet equation

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}. \tag{2.7}$$

Now we shall analize the above relations using the bilinear formalism. First we introduce four dependent variables $f^*$, f, $g^*$ and g, where $*$ denotes complex conjugate, and constuct the tangent vector $\hat{t} = (t_1, t_2, t_3)$

$$t_1 + i\,t_2 = \frac{2f^*g}{f^*f + g^*g}, \tag{2.8}$$

$$t_3 = \frac{f^*f - g^*g}{f^*f + g^*g}. \tag{2.9}$$

Note that $$t_1^2 + t_2^2 + t_3^2 = 1. \tag{2.10}$$

Then the Seret-Frenet equation gives that $\psi$ is expressed with $f^*$, f, $g^*$ and g as follows.

$$\psi = \kappa \exp[i\int_{-\infty}^{x} \tau(t,x')dx'] \tag{2.11}$$

$$= \frac{2D_x g\cdot f}{f^*f + g^*g}, \tag{2.12}$$

provide that the following condition on $f^*$, f, $g^*$ and g are satisfied

$$D_x(f^*\cdot f + g^*\cdot g) = 0. \tag{2.13}$$

On the other hand, we know that the nonlinear Schrödinger equation is transformed into the bilinear form

$$(iD_t + D_x^2)G\cdot F = 0, \tag{2.14a}$$

$$D_x^2 F\cdot F = 2G^*G, \tag{2.14b}$$

though the transformation

$$\psi = G/F, \tag{2.15}$$

where F is a real function.

The equation (2.12) suggests the composite structure of the soliton solution to the nonlinear Schrödinger equation, Actually we have proved that F and G defined by

$$F^2 = f^* f + g^* g, \tag{2.16a}$$

$$GF = D_x g \cdot f, \tag{2.16b}$$

satisfy eq.(2.14) provided that $f^*$, f, $g^*$ and g satisfy the following bilinear equation

$$D_x(f^* \cdot f + g^* \cdot g) = 0, \tag{2.17a}$$

$$(iD_t + D_x^2)(f^* \cdot f - g^* \cdot g) = 0, \tag{2.17b}$$

$$(iD_t + D_x^2) f \cdot g^* = 0. \tag{2.17c}$$

Furthermore we find the classical Heisenberg ferromagnet equation in the continuum limit

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} \tag{2.18}$$

is also transformed into eq.(2.17) through the transformation

$$\vec{S} = (S_1, S_2, S_3), \tag{2.19a}$$

$$S_1 + iS_2 = \frac{2f^* g}{f^* f + g^* g}, \tag{2.19b}$$

$$S_3 = \frac{f^* f - g^* g}{f^* f + g^* g}. \tag{2.19c}$$

The bilinear equation (2.17) plays the fundamental role in the nonlinear Schrödinger equation and in the classical Heisinberg ferromagnet.

## §3. Pohlmeyer-Lund-Regge-Getmanov Equation

In this section, we consider the Pohlmeyer-Lund-Regge equation

$$\theta_{xt} - \sin\theta\cos\theta - \frac{\sin\theta}{\cos^3\theta}\beta_x\beta_t = 0, \tag{3.1a}$$

$$(\beta_t \tan^2\theta)_x + (\beta_x \tan^2\theta)_t = 0, \tag{3.1b}$$

which is transformed into the Getmanov equation

$$\varphi_{xt} + \frac{\varphi_x \varphi_t \varphi^*}{1-\varphi^*\varphi} - \varphi(1-\varphi^*\varphi) = 0 \tag{3.2}$$

through the transfomation

$$\varphi = \sin\theta\, e^{i\beta}. \tag{3.3}$$

The Pohlmeyer-Lund-Regge equation has been found through the geometrical consideration of the sine-Gordon equation and is known to be transformed into the inverse scattering form.

On the other hand, Getmanov transformed eq.(3.2) into the trilinear equation

$$D_x D_t F\cdot F = 2g^*g, \tag{3.4a}$$

$$F[(D_x D_t - 1)g\cdot F] = (1/2)g^* D_x D_t g\cdot g \tag{3.4b}$$

through the transformation

$$\varphi = g/F, \quad F \text{ being real}. \tag{3.5}$$

He found 2-soliton solution to it and conjectured N-soliton solutions.

We shall show in the appendix that F of eq.(3.5) has the same structure as that of eq.(2.16)

$$F^2 = f^*f + g^*g \tag{3.6}$$

and the trilinear equations are transformed into the bilinear equation

$$D_x(f^*\cdot f + g^*\cdot g) = 0, \tag{3.7a}$$

$$D_x D_t(f^*\cdot f - g^*\cdot g) + 2g^*g = 0, \tag{3.7b}$$

$$(D_x D_t - 1)f\cdot g^* = 0. \tag{3.7c}$$

The bilinear equation $D_x(f'\cdot f + g'\cdot g) = 0$ suggests that the quantity defined by

$$\psi = \frac{2D_x g\cdot f}{f^*f + g^*g} \tag{3.8}$$

would play a similar role to that in the nonlinear Schrödinger equation.

In fact, $\psi$ defined by eq.(3.8) is found to satisfy the equation

$$\psi_{xt} = \psi(1-|\psi_t|^2)^{1/2} \qquad (3.9)$$

which is much simpler than the Pohlmeyer-Lund-Regge-Getmanov equation.

The Pohlmeyer-Lund-Regge equation reduces to the sine-Gordon equation

$$\theta_{xt} = \sin\theta\cos\theta \qquad (3.10)$$

if $\beta$ =0.

It is known that the sine-Gordon equation is transformed into the bilinear form

$$D_x D_t (f\cdot f - g\cdot g) = 0, \qquad (3.11a)$$

$$(D_x D_t - 1) f\cdot g = 0, \qquad (3.11b)$$

through the transformation

$$\theta = 2\tan^{-1}(g/f). \qquad (3.12)$$

For real f and g, eq.(3.7 b) and (3.7 c) is the same as eqs.(3.11 a), (3.11 b) except the second term $2g^*g$ in eq.(3.7b). This suggests us another generalization of the sine-Gordon equation to the complex one.

We introduce a set of bilinear equation

$$D_x(f^*\cdot f + g^*\cdot g) = 0, \qquad (3.13a)$$

$$D_x D_t (f^*\cdot f - g^*\cdot g) = 0, \qquad (3.13b)$$

$$(D_x D_t - 1) f^*\cdot g = 0, \qquad (3.13c)$$

and construct a compoex sine-Gordon equation through the dependent variable transformation

$$f = \exp(\rho)\cos\theta\,\exp(i\alpha), \qquad (3.14a)$$

$$g = \exp(\rho)\sin\theta\,\exp(i\beta), \qquad (3.14b)$$

where $\rho$, $\theta$, $\alpha$ and $\beta$ are real functions of x and t.

Substituting eq.(3.14) into eqs.(3.13) and eliminating $\rho$ and $\alpha$ we obtain

$$\theta_{xt} - \frac{1}{4}\sin 4\theta - \frac{\sin\theta}{\cos^3\theta}\beta_x\beta_t = 0, \qquad (3.15a)$$

$$(\beta_t \tan^2\theta)_x + (\beta_x \tan^2\theta)_t = 0, \qquad (3.15b)$$

which is the same equation as the Pohlmeyer-Lund-Regge equation except the second term in eq.(3.15 a) where we have $(1/4)\sin 4\theta$ in stead of $(1/2)\sin 2\theta$. The difference is crutial in the nonlinear evolution equation. Any scale transformation never transform eq.(3.1) into eq.(3.15).

Hereafter we call eq.(3.15) "complex sine-Gordon equation."

Now we have two nonlinear evolution equations as generalizations of the sine_Gordon equation. One is the Pohlmeyer-Lund-Regge equation and another is the complex sine-Gordon equation. We met with the similar situations before. The first case is concerned with the higher order KdV equation. There are three types of fifth order KdV equation, the fifth order KdV equation of Lax, the Sawada-Kotera equation and Kaup's equation. They differ only in the coefficients of nonlinear terms. The second case is concerned with the model equations for shallow water waves. Among them the fifth order KdV equation of Lax, Kaup's equation and one of the model equatio for shallow water waves are obtained through the inverse scattering formalism. The others are obtained through the bilinear formalism.

One of other generalization of the nonlinear evolution equations is to construct the differnce-differnce equations. Difference analogues of the nonlinear evolution equations were first constructed by using the inverse scattering formalism, and then by using the bilinear formalism. We have constructed a discrete sine-Gordon equation using the bilinear equation (3.10).

Taking the similarity between eqs.(3.7) and (3.11) into account, it is possible to construct difference analogues of the Pohlmeyer-Lund-Regge equation and of the complex sine-Gordon equation. We shall describe them in the forth coming paper.

## §4. Landau-Lifshitz Equation

The Landau-Lifshitz equation

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \vec{S} \times J \vec{S} \tag{4.1}$$

where

$$\vec{S} = (S_1, S_2, S_3), \quad \|\vec{S}\| = 1, \tag{4.2}$$

$$J = \mathrm{diag}(J_1, J_2, J_3), \quad J_1 < J_2 < J_3, \tag{4.3}$$

describes nonlinear spin waves in a ferromagnet propagating in a direction orthogonal to the anisotropic axis.

Sklyanin and Borovik have succeed in representing eq.(4.1) as a compatibility condition

$$L_t - M_x + [L, M] = 0, \tag{4.4}$$

for the set of two equations for 2×2 matrices $\phi(x,t;\lambda)$

$$\phi_x = L\phi, \quad L = L(\vec{S};\lambda) \tag{4.5a}$$

$$\phi_t = M\phi, \quad M = M(\vec{S};\lambda) \tag{4.5b}$$

where $\lambda$ is the spectral parameter.

Bogdan and Kovalev have succeed in transforming eq.(4.1) into the trilinear form

$$f^*f\{[iD_t + D_x^2 - a(1+b)]g\cdot f\} + g^*\{D_x^2 g\cdot g - a(1-b)f\cdot f\} = 0, \tag{4.6a}$$

$$g^*\{[-iD_t + D_x^2 - a(1+b)]g\cdot f\} + f^*\{D_x^2 f\cdot f - a(1-b)gg\} = 0, \tag{4.6b}$$

through the transformation

$$S_1 + iS_2 = \frac{2f^*g}{f^*f + g^*g}, \tag{4.7a}$$

$$S_3 = \frac{f^*f - g^*g}{f^*f + g^*g}, \tag{4.7b}$$

$$a = (J_3 - J_1)/2, \tag{4.8a}$$

$$b = (J_3 - J_2)/(J_3 - J_1). \tag{4.8b}$$

They have obtained 2-soliton solution and conjectured N-soliton solutions.

We have transformed eq.(4.1) into the bilinear form.

$$D_x(f^* \cdot f + g^* \cdot g) = 0, \tag{4.9a}$$

$$(iD_t + D_x^2)(f^* \cdot f - g^* \cdot g) = 0, \tag{4.9b}$$

$$(iD_t + D_x^2) f \cdot g^* - a(f^*g + fg^*) + ab(f^*g - fg^*) = 0. \tag{4.9c}$$