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An Analysis of the Painlevé Equations by Bilinearization

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1. Introduction

Recently, in a series of papers[1-6] the present author has shown that the solutions expressing interactions between solitons and ripples can be obtained for a wide range of soliton equations by starting with their bilinear forms. Moreover, it has been also shown[7-9] that some special examples of the Painlevé equations can be analysed by using their bilinear forms. In the present note, we shall briefly review such results by taking the Hirota equation[10] and a special form of the fourth Painlevé equation (P_{IV}, for short) as examples. It should be noted here that such results have deep relations to the celebrated theory of the monodromy preserving deformation of the ordinary differential equations, which has recently been developed strikingly by M. Sato, Y. Sato, Miwa, Jimbo, Ueno and others.[11]

2. The Hirota Equation

In this section, we are concerned with the Hirota equation[10] of the following form

$$iu_t+i3\alpha|u|^2u_x+\beta u_{xx}+i\gamma u_{xxx}+\delta|u|^2u=0, \tag{2.1}$$

where $u=u(t,x)$, the subscripts denote partial differentiations and $\alpha,\beta,\gamma,$ and δ are real constants satisfying the relation

$$\alpha\beta=\gamma\delta \tag{2.2}$$

It is known[10] that eq. (2.2) can be transformed into a bilinear form

$$(iD_t+\beta D_x^2+i\gamma D_x^3)g\cdot f=0 \tag{2.3.a}$$

$$\gamma D_x^2 f\cdot f=\alpha gg^*, \tag{2.3.b}$$

by the dependent variable transformation

$$u=g/f, \tag{2.4}$$

where f is a real function. From eq. (2.3.b), it follows that

$$|u|^2=\frac{\alpha}{2\gamma}(\log f)_{xx}. \tag{2.5}$$

The solution of eq. (2.3) expressing interactions between solitons and ripples can be obtained in the following form

$$f=1+\sum_{n=1}^{\infty} f_{2n} \tag{2.6.a}$$

$$g=\sum_{n=1}^{\infty} g_{2n-1}, \tag{2.6.b}$$

where

$$f_{2n}=\frac{1}{(n!)^2}\int_\Gamma\cdots\int_\Gamma\int_{\Gamma*}\cdots\int_{\Gamma*}\prod_{1\leq h<j\leq 2n}\psi_n(h,j)$$

$$\times\exp[\sum_{j=1}^{n}(\phi_j+\tilde{\phi}_{n+j})]\prod_{j=1}^{n}d\tau(k_j)d\tau^*(k^*_{j+n}) \qquad (2.7.a)$$

and

$$g_{2n-1}=\frac{1}{n!(n-1)!}\int_\Gamma\cdots\int_\Gamma\int_{\Gamma*}\cdots\int_{\Gamma*}\prod_{1\leq h<j\leq 2n-1}\psi_n(h,j)$$

$$\times\exp[\sum_{j=1}^{n}(\phi_j+\tilde{\phi}_{n+j})+\phi_n][\prod_{j=1}^{n-1}d\tau(k_j)d\tau^*(k_{n+j}{}^*)]d\tau(k_n). \qquad (2.7.b)$$

Here,

$$\phi_j=k_jx-\Omega_jt,\quad \Omega_j=-i\beta k_j^2+\gamma k_j^3,\quad (j=1,2,\cdots,n), \qquad (2.8.a)$$

$$\tilde{\phi}_j=k_j^*x-\Omega_j^*t,\quad \Omega_j^*=i\beta k_j^{*2}+\gamma k_j^{*3},\quad (j=n+1,n+2,\cdots,2n), \qquad (2.8.b)$$

and

$$\exp\,\psi_n(h,j)=\begin{cases}\frac{\alpha}{2\gamma}(k_h+k_j^*)^{-2}, & (1\leqq h\leqq n,\ n+1\leqq j\leqq 2n)\\ \frac{2\gamma}{\alpha}(k_h-k_j)^2, & (1\leqq h<j\leqq n)\\ \frac{2\gamma}{\alpha}(k_h^*-k_j^*)^2, & (n+1\leqq h<j\leqq 2n)\end{cases} \qquad (2.8.c)$$

The measure $d\tau(k)$ is defined by $\int_\Gamma d\tau(k)=\int_\Gamma a(k)dk$, where $a(k)$ is an arbitrary function of k satisfying suitable conditions and Γ

is a contour in a left half of the complex k-plane going from $k=-i\infty$ to $k=i\infty$.

We can see that the function f and g defined be eq. (2.6) really satisfy eq.(2.3) from the facts that eq. (2.3) has the N-soliton solution and the functions f_{2n} and g_{2n-1} enjoy the following relations

$$\sum_{j=0}^{n}(iD_t+\beta D_x^2+i\gamma D_x^3)g_{2(n-j)+1}\cdot f_{2j}=0, \qquad (2.9.2n+1)$$

$$\sum_{j=0}^{n}(\gamma D_x^2 f_{2(n-j)}\cdot f_{2j}-\alpha g_{2(n-j)-1}\cdot g^*_{2(j+1)-1})=0, \qquad (2.9.2n)$$

for $n=0,1,2,\cdots$, where $g_{-1}=0$. We note here that the function f can be rewritten in a Fredholm determinant form

$$f=1+\sum_{n=1}^{\infty}\frac{\alpha}{2\gamma(n!)^2}\int_x^{\infty}\cdots\int_x^{\infty}\det(\Phi_n\Phi_n^*)\prod_{j=1}^{n}dx_j dy_j$$

$$=1+\sum_{n=1}^{\infty}\frac{\alpha}{2\gamma n!}\int_x^{\infty}\cdots\int_x^{\infty}\det(\Psi_n)\prod_{j=1}^{n}dx_j, \qquad (2.10)$$

where Φ_n and Ψ_n are n×n matrices defined respectively from the function $F(x)=\int_\Gamma \exp(kx-\Omega t)d\tau(k)$ as follows: the h-j element of Φ_n is $F(x_h+y_j)$ and that of Ψ_n is $\int_x^{\infty}F(x_h,y)F^*(y,x_j)dy$. It is easily seen that the function f is the Fredholm determinant of the following integral equation

$$K(x,y)+F(x+y)+\int_x^{\infty}\int_x^{\infty}K(x,s_1)F(s_1,s_2)F^*(s_2,y)ds_1ds_2=0. \qquad (2.11)$$

3. The Fourth Painlevé Equation

We now show that using the result obtained in the preceeding section, we can derive an one parameter family of solutions for a special form of P_{IV}. To this end, we note that the nonlinear Schrödinger equation

$$iu_t+2|u|^2u_x+u_{xx}=0, \tag{3.1}$$

which is obtained form eq. (2.1) by putting $\alpha=\gamma=0$ and $\beta=\delta/2=2$, reduces to a special form of P_{IV}

$$w''=(2w)^{-1}(w')^2-(32w)^{-1}-2w(3w^2-zw+w/16) \tag{3.2}$$

by the similarity transformation[12)]

$$u(t,x)=v(z),\ w(z)=\frac{1}{2i}[\log(e^{iz^2/4}v(z)/v^*(z))]'. \tag{3.3}$$

Here, the prime denotes the differentiation with respect to $z=xt^{-1/2}$ and in the following, we shall refer to eq. (3.2) as P'_{IV}.

In the first place, we show that the bilinear form of P'_{IV} can be derived from that of eq. (3.1). For the purpose, we put

$$f(t,x)=\tau_1(z) \tag{3.4.a}$$

$$g(t,x)=t^{-1/2}\tau_2(z) \tag{3.4.b}$$

in eq. (2.3) with $\alpha\to 0$, $\gamma\to 0$, $\alpha/\gamma\to 2$, and $\beta=\delta/2\to 1$:

$$(iD_t+D_x^2)g\cdot f=0 \tag{3.5.a}$$

$$D_x^2 f\cdot f=2gg^*. \tag{3.5.b}$$

Then, eq. (3.5) is transformed into

$$[i(zD_z+I)-2D_z^2]\tau_2\cdot\tau_1=0 \tag{3.6.a}$$

$$D_z^2\tau_1\cdot\tau_2=2\tau_2^*\tau_2 \tag{3.6.b}$$

and eq. (3.3) into

$$w(z)=\frac{1}{2i}[\log e^{iz^2/4}\tau_2(z)/\tau_2^*(z)]'. \tag{3.7}$$

Thus we found that eq. (3.6) is the bilinear form of P_{IV}' and is connected with P_{IV}' through eq. (3.7).

We now construct an one parameter family of solutions for P_{IV}'. For the purpose, we note that if a solution of eq. (3.5) enjoys the conditions (3.4.a,b), it is also a soltions of eq. (3.6). Having this in mind, put

$$d\tau(k)=dk, \tag{3.8.a}$$

$$\phi_j=i(k_jx-k_j^2t),\ (j=1,2,\cdots,n) \tag{3.8.b}$$

$$\tilde{\phi}_j=-i(k_jx-k_j^{*2}t),\ (j=n+1,n+2,\cdots,2n) \tag{3.8.c}$$

$$\exp\psi_n(h,j)=\begin{cases}-(k_h-k_j^*)^{-2}, & (1\leqq h\leqq n,\ n+1\leqq j\leqq 2n)\\ -(k_h-k_j)^2, & (1\leqq h<j\leqq n)\\ -(k_h^*-k_j^*)^2, & (n+1\leqq h<j\leqq 2n)\end{cases} \tag{3.8.d}$$

and $\int_\Gamma d\tau(k)=\int_{-\infty}^{\infty}dk$ in eq. (2.7). Then, we can see that

$$f_{2n}(t,x)=\tau_{1,2n}(z) \tag{3.9.a}$$

$$g_{2n-1}(t,x)=t^{-1/2}\tau_{2,2n-1}(z) \tag{3.9.b}$$

so that the functions f and g enjoy the conditions (3.4.a,b), where

$$\tau_{1,2n}(z)=f_{2n}(t=1,z)\,|_{d\tau(k)=dk} \tag{3.10.a}$$

$$\tau_{2,2n-1}(z)=g_{2n-1}(t=1,z)\,|_{d\tau(k)=dk}. \tag{3.10.b}$$

Thus, we found that the functions

$$\tau_1(z)=1+\sum_{n=1}^{\infty}a^{2n}\tau_{1,2n}(z) \tag{3.11.a}$$

and

$$\tau_2(z)=\sum_{n=1}^{\infty}a^{2n-1}\tau_{2,2n-1}(z) \tag{3.11.b}$$

become a solution of eq. (3.6), where a is an arbitrary real parameter.

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