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On stationary axially symmetric Einstein equations

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In this note we apply the soliton theory to the problem of finding exact solutions of the stationary axially symmetric Einstein equations. Several works have been done for this purpose. For example, the existence of inverse scattering formulae manifested the relations of the equations to the soliton equations [1],[2],[3]. A certain inverse scattering problem was actually solved to give the soliton type solution [4]. On the other hand, some Bäcklund transformations were presented without deriving exact solutions [5],[6]. The present work gives a Bäcklund transformation and Kasner type solutions for the stationary axially symmetric Einstein equations.

1. Spin representation and reduced system of Einstein equations

As is well known, the stationary axially symmetric gravitational field can be found from the solution of the Ernst's equation [7]

$$(\mathrm{Re}E)\Delta E = \nabla E\cdot\nabla E, \tag{1}$$

where $E = f + i\psi$. f and ψ are functions of ρ, z only, ∇ and Δ are the gradient and Laplacian operators in the three-dimensional Euclidean space, respectively.

We define a three-dimensional vector S; $S = {}^t(S_1, S_2, S_3)$, as follows

$$S_1 = \frac{1 + f^2 + \psi^2}{2f}, \quad S_2 = \frac{1 - f^2 - \psi^2}{2f}, \quad S_3 = \frac{\psi}{f}.$$

REMARK 1. The vector S satisfies a pseudonorm constraint condition

$$(S \cdot S) = -(S_1)^2 + (S_2)^2 + (S_3)^2 = -1.$$

Using this property and the Lagrangian function for (1), we find that that the equation (1) can be rewritten as

$$\Delta S = (\nabla S \cdot \nabla S) S. \tag{2}$$

This is a spin representation of the stationary axially symmetric Einstein equations. The spin representations for the other soliton equations are given in [8],[9].

Instead of (ρ, z), we set complex conjugate coordinates $(\eta, \bar{\eta})$;

$$\eta = \frac{1}{2}(\rho + iz), \quad \bar{\eta} = \frac{1}{2}(\rho - iz).$$

Next, we define complex functions A and B in terms of the derivatives of S as follows

$$\begin{gathered} (S_\eta \cdot S_\eta) = A\bar{B}, \quad (S_{\bar{\eta}} \cdot S_{\bar{\eta}}) = \bar{A}B, \\ (\nabla S \cdot \nabla S) = (S_\eta \cdot S_{\bar{\eta}}) = \frac{1}{2}\{|A|^2 + |B|^2\}. \end{gathered} \tag{3}$$

If the metric is asymptotically flat, then the functions A and B go to zero at infinity.

REMARK 2. By the use of the original unknown functions f and ψ, A and B can be denoted as

$$A = (\log f)_{\eta} + i\psi_{\eta} f^{-1}, \quad B = (\log f)_{\bar{\eta}} + i\psi_{\bar{\eta}} f^{-1}. \tag{4}$$

We derive a reduced system for A and B. We obtain

$$\begin{aligned} &2A_{\bar{\eta}} + A(\bar{A} - B) + (A + B)(\eta + \bar{\eta})^{-1} = 0, \\ &2B_{\eta} + B(\bar{B} - A) + (A + B)(\eta + \bar{\eta})^{-1} = 0. \end{aligned} \tag{5}$$

We call equations (5) and their complex conjugates the reduced system of the stationary axially symmetric Einstein equations, and write them R_1, R_2, $\bar{R}_1$, $\bar{R}_2$, respectively.

2. Bäcklund transformation for reduced system.

The reduced system $R_1, R_2, \cdots$ is a compatibility condition of linear differential operators. Making use of the variables η, $\bar{\eta}$ and a constant $\varepsilon \in R$, we define a function ζ as

$$\zeta^2 = (\varepsilon + i\bar{\eta})(\varepsilon - i\eta)^{-1} \tag{6}$$

We have the following theorem.

THEOREM 1. Define the linear differential operators

$$\begin{aligned} L &= \begin{pmatrix} \partial/\partial\eta & 0 \\ 0 & \partial/\partial\eta \end{pmatrix} + \frac{A - \bar{B}}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\zeta}{2}\begin{pmatrix} 0 & A - (\eta + \bar{\eta})^{-1} \\ \bar{B} - (\eta + \bar{\eta})^{-1} & 0 \end{pmatrix}, \\ \bar{L} &= \begin{pmatrix} \partial/\partial\bar{\eta} & 0 \\ 0 & \partial/\partial\bar{\eta} \end{pmatrix} + \frac{\bar{A} - B}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\zeta^{-1}}{2}\begin{pmatrix} 0 & \bar{A} - (\eta + \bar{\eta})^{-1} \\ B - (\eta + \bar{\eta})^{-1} & 0 \end{pmatrix}, \end{aligned} \tag{7}$$

In and only if the functions A and B satisfy the reduced system, the operators L and $\bar{L}$ are compatible with each other, that is,

$$[\, L \,, \bar{L} \,] = 0. \tag{8}$$

REMARK 3. For the reduced system R_1, R_2, the operator L and $\bar{L}$ give an inverse scattering formula

$$L\begin{vmatrix}\phi_1\\ \phi_2\end{vmatrix} = 0, \quad \bar{L}\begin{vmatrix}\phi_1\\ \phi_2\end{vmatrix} = 0, \tag{9}$$

where ϕ_1 and ϕ_2 are eigenfunctions.

Next, we discuss a Bäcklund transformation for the reduced system. We obtain the following main theorem.

THEOREM 2. Define the function $\Phi(\eta,\bar{\eta})$ by $\Phi = \phi_1\phi_2^{-1}$, where Φ is not equal to $-\zeta^{\pm 1}$. If the functions A and B satisfy the reduced system, then $A^{(1)}$ and $B^{(1)}$, defined by the transformations

$$\begin{aligned} A^{(1)} &= (1+\zeta\Phi)(\eta+\bar{\eta})^{-1} - (\Phi+\zeta\Phi^2)(\zeta+\Phi)^{-1}A, \\ B^{(1)} &= (1+\zeta\Phi)\{\zeta\Phi(\eta+\bar{\eta})\}^{-1} - (1+\zeta\Phi)(\zeta\Phi+\Phi^2)^{-1}B \end{aligned} \tag{10}$$

also satisfy the reduced system.

REMARK 4. If and only if $A = B = 0$, $\Phi = -\zeta^{\pm 1}$ are the solutions of (9). Simultaneously, the second terms of the right-hand sides of (10) vanish. Thus we may consider that the transformation (10) includes the case $\Phi = -\zeta^{\pm 1}$.

REMARK 5. Equations (10) give a Bäcklund transformation for the reduced system. A similar transformation has been derived by [5].

A new solution $E^{(1)} = f^{(1)} + i\psi^{(1)}$ of equation (1) is obtained from $A^{(1)}$ and $B^{(1)}$. We have

PROPOSITION 1. Let $A^{(1)}$ and $B^{(1)}$ satisfy the reduced system. If there are some functions $f^{(1)}$ and $\psi^{(1)}$ satisfying

$$d \log f^{(1)} = \frac{A^{(1)} + \bar{B}^{(1)}}{2} d\eta + \frac{\bar{A}^{(1)} + B^{(1)}}{2} d\bar{\eta},$$
$$d\psi^{(1)} = \frac{A^{(1)} - \bar{B}^{(1)}}{2i} f^{(1)} d\eta - \frac{\bar{A}^{(1)} - B^{(1)}}{2i} f^{(1)} d\bar{\eta}. \tag{11}$$

then $f^{(1)}$ and $\psi^{(1)}$ are another solution of equation (1).

We have completed an algorithm to construct exact solutions of the field equation.

3. Concrete solutions: generelization of Kasner solutions.

In this section, we consider an application of the Bäcklund transformation. We use the Minkowski metric as a known solution, i. e.,

$$ds^2 = - dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2,$$

in the cylindrical coordinates (ρ, z, ϕ). The complex potential E and the functions A, B are written as $E = 1$, $A = B = 0$. Then, the function $\Phi(\eta,\bar{\eta})$ satisfies

$$\Phi_\eta = \pm\{2(\eta + \bar{\eta})\}^{-1}\sqrt{\frac{\varepsilon + i\bar{\eta}}{\varepsilon - i\eta}}\,(1 - \Phi^2),$$
$$\Phi_{\bar{\eta}} = \pm\{2(\eta + \bar{\eta})\}^{-1}\sqrt{\frac{\varepsilon - i\eta}{\varepsilon + i\bar{\eta}}}\,(1 - \Phi^2).$$

The resulting solution is written in the form:

$$\Phi = (P + \delta)(P - \delta)^{-1}, \quad (- P + 3\delta)(P - \delta)^{-1},$$

where δ is a nonnegative constant and P is defined by

$$P = \left|\frac{\sqrt{\varepsilon - i\eta} \mp \sqrt{\varepsilon + i\bar{\eta}}}{\sqrt{\varepsilon - i\eta} \pm \sqrt{\varepsilon + i\bar{\eta}}}\right|.$$

We shall restrict ourselves to the case $\delta = 0$. By the

Bäcklund transformation $\Phi = +1$, and $\Phi = -1$, which are denoted as Φ_+, Φ_-, respectively, we generate some solutions of equation (1). Under the transformation Φ_+, we have

$$A^{(1)} = (1 + \zeta)(\eta + \bar{\eta})^{-1}, \; B^{(1)} = (1 + \zeta^{-1})(\eta + \bar{\eta})^{-1},$$

$$f^{(1)} = \gamma^{(1)}(\eta + \bar{\eta})\,|(\zeta - 1)(\zeta + 1)^{-1}|, \; \psi^{(1)} = 0,$$

where $\gamma^{(1)}$ is a positive constant. By the transformation Φ_-, new solutions $A^{(2)}$, $B^{(2)}$ of reduced system are obtained from $A^{(1)}$, $B^{(1)}$ as

$$A^{(2)} = -2\zeta(\eta + \bar{\eta})^{-1}, \; B^{(2)} = -2\zeta^{-1}(\eta + \bar{\eta})^{-1}.$$

Then, we get

$$f^{(2)} = \gamma^{(2)}\,|(\zeta + 1)(\zeta - 1)^{-1}|^2, \; \psi^{(2)} = 0,$$

where $\gamma^{(2)}$ is a positive constant. This is the result of the product transformation $\Phi_- \circ \Phi_+$.

REMARK 6. By the product transformation $\Phi_+ \circ \Phi_+$ or $\Phi_- \circ \Phi_-$, the function f does not become complicated at all. By using Φ_+ and Φ_- successively, a series of solution $f^{(1)}$, $f^{(2)}$, $\cdots$ can be derived.

If the transformations Φ_+ and Φ_- are repeated N times, the solution $f^{(N)}$ reads

$$\begin{aligned} f^{(N)} &= \gamma^{(N)}(\eta + \bar{\eta})\,|(\zeta - 1)(\zeta + 1)^{-1}|^{(N+1)/2} \quad (N\text{:odd}), \\ f^{(N)} &= \gamma^{(N)}\,|(\zeta + 1)(\zeta - 1)^{-1}|^{(N+2)/2} \quad (N\text{:even}), \end{aligned} \tag{12}$$

where $\gamma^{(N)}$ are positive constants. It is not hard to represent

(12) in (ρ, z) coordinates. We have

$$f^{(N)} = \gamma^{(N)} \rho \left| \{\sqrt{(2\varepsilon + z)^2 + \rho^2} \mp (2\varepsilon + z)\}\rho^{-1} \right|^{(N+1)/2} \quad (N\text{:odd}),$$

$$f^{(N)} = \gamma^{(N)} \left| \{\sqrt{(2\varepsilon + z)^2 + \rho^2} \pm (2\varepsilon + z)\}\rho^{-1} \right|^{(N+2)/2} \quad (N\text{:even}), \tag{13}$$

where the double signs $\mp$ (N:odd) and $\pm$ (N:even) correspond to $\zeta = \pm(\varepsilon + i\bar{\eta})^{1/2}(\varepsilon - i\eta)^{-1/2}$.

We here set $\gamma^{(N)} = (4|\varepsilon|)^{(N+1)/2}$ (N:odd), $\gamma^{(N)} = (4|\varepsilon|)^{\mp(N+2)/2}$ (N:even), and take the limit $\varepsilon \to \infty$. The solution (13) are now

$$f^{(N)} = \rho^{1 \pm (N+1)/2} \quad (N\text{:odd}),$$

$$f^{(N)} = \rho^{\mp (N+2)/2} \quad (N\text{:even}). \tag{14}$$

Finally, we complete the metric with respect to (14). The remaining unknown functions can be determined to be

$$ds^2 = -\rho^{1 \pm (N+1)/2} dt^2 + \rho^{(N+3)(N-1)/8} \cdot (d\rho^2 + dz^2) + \rho^{1 \mp (N+1)/2} d\phi^2 \quad (N\text{:odd}),$$

$$ds^2 = -\rho^{(N+2)/2} dt^2 + \rho^{(N+2)(N+2 \pm 4)/8} \cdot (d\rho^2 + dz^2) + \rho^{2 \pm (N+2)/2} d\phi^2 \quad (N\text{:even}).$$

By suitable coordinate transformations, they are reduced to

$$ds^2 = -\rho^{2\lambda} dt^2 + d\rho^2 + \rho^{2\mu} dz^2 + \rho^{2\nu} d\phi^2, \tag{15}$$

where

$$\lambda = 4(2 \pm N \pm 1)/C, \quad \mu = (N^2 + 2N - 3)/C,$$

$$\nu = 4(2 \mp N \mp 1)/C, \quad C = N^2 + 6N + 2, \quad (N\text{:odd}),$$

$$\lambda = \mp 4(N + 2)/D, \quad \mu = \{N^2 + (4 \pm 4)N + 4 \pm 8\}/D,$$

$$\nu = 4(4 \pm N \pm 2)/D, \quad D = N^2 + (4 \pm 4)N + 4 \pm 24, \quad (N\text{:even}),$$

$$(\lambda + \mu + \nu = 1, \quad \lambda\mu + \mu\nu + \nu\lambda = 0).$$

The solutions (15) are nothing but the polynomial solutions discovered by Kasner [10], while the solutions (13) are new generalizations of Kasner type solutions.

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