A Note on 2-fold Branched Covers of $S^3$

Taizo KANENOBU
Kobe University

1. Introduction

If a knot $K \subset S^3$ is amphicheiral, then the 2-fold covering space branched over $K$ is symmetric (has an orientation reversing diffeomorphism). In this note we show that the converse is not true; it solves Problem 1.23 of [K], which is suggested by Montesinos [Ml].

Theorem. There exist non-amphicheiral knots whose 2-fold branched covering spaces are symmetric.

In Section 2, we will construct a composite knot satisfying the properties of Theorem, and in Section 3, a prime knot.

2. A Composite Knot

Let $K_1$ and $K_2$ be two knots. If there exists an orientation preserving homeomorphism of $S^3$ which maps $K_1$ onto $K_2$, then we will write $K_1 \cong K_2$.

Lemma 1. Let $K_1$ and $K_2$ be two prime knots such that $K_1 \cong r(K_2)$ where $r$ is an orientation reversing homeomorphism of $S^3$. Then the
composite knot $K_1 \# K_2$ is amphicheiral if and only if both $K_1$ and $K_2$ are amphicheiral.

Proof. If $K_1 \# K_2$ is amphicheiral, then $K_1 \# K_2 \approx r(K_1 \# K_2) = r(K_1) \# r(K_2)$. It follows from the unique decomposition theorem [Sc] and $K_1 \not\approx r(K_2)$ that $K_1 \approx r(K_1)$ and $K_2 \approx r(K_2)$. The converse is clear.

Let $K_1$ be the pretzel knot (see [Tr] or [P]) of type $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n)$, denoted by $K(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n)$, where $n$ (\geq 5) is an odd integer and $p_1, p_2, \ldots, p_n \ (\geq 3)$ are distinct and odd integers. Let $K_2 = K(p_1, \ldots, p_j, \ldots, p_i, \ldots, p_n)$, where $i \neq j$.

By the theorem in §2 of [Ml], the 2-fold covering spaces of $S^3$ branched over $K_1$ and $K_2$ are the same Seifert fiber space $(000|0; (p_1, 1), (p_2, 1), \ldots, (p_n, 1))$ [Se], or the manifold with the following surgery presentation (see [R, Chapter 9]):

This manifold is prime by Theorem 7.1 and Lemma 10.2 of [Wl]. Hence by the main result of [W2], $K_1$ and $K_2$ are prime knots. By the classification theorem of pretzel knots [P, p.56], we have $K_1 \not\approx K_2$. Since the signatures of $K_1$ and $K_2$ are both $n-1 \ (\neq 0)$ [P, p.71], $K_1$ and $K_2$ are non-amphicheiral [R, p.217]. It follows from Lemma 1 that $K_1 \# r(K_2)$ is non-amphicheiral.
On the other hand, the 2-fold covering space of $S^3$ branched over $K_1 \# r(K_2)$ is clearly symmetric. Therefore $K_1 \# r(K_2)$ is the desired composite knot.

Remark. We can also construct such a composite knot by using the examples of prime knots in [BGM] and [Ta] instead of pretzel knots.

3. A Prime Knot

Let $M$ be the manifold which is obtained by Dehn surgery in the link $L$ according to the diagram of Fig. 1. Since the figure-eight knot, which is the sublink of $L$, is amphicheiral, it is easily verified that $M$ is symmetric. The link $L$ is strongly-invertible because the symmetry with respect to the axis $E$ leaves $L$ invariant. Using the method of [M2], it can be shown that $M$ is a 2-fold covering space branched over the knot $K$ of Fig. 2. Because the signature of $K$ is 16, $K$ is non-amphicheiral. The following fact will be proved in Section 4.

Lemma 2. $K$ is a prime knot.

4. Proof of Lemma 2

We will prove Lemma 2 by the method of Kirby and Lickorish [KL]. A tangle is a set of disjoint two arcs properly embedded in a 3-ball. The tangle as shown in Fig. 3 will be called a trivial tangle. A tangle $T$ in a 3-ball $B$ is prime if it has the following properties.

(i) Any 2-sphere in $B$, which meets $T$ transversely in two points, bounds in $B$ a ball meeting $T$ in an unknotted spanning arc.
(ii) The arcs of $T$ cannot be separated by a disc properly embedded in $B$.

The knot $K$ is constructed by the tangles $T_1$ and $T_2$ as shown in Fig. 4. To establish that $K$ is prime, we have only to show that the tangles $T_1$ and $T_2$ are prime. If $T_i$ ($i = 1, 2$) did not satisfy property (i) or property (ii), then there would be a ball meeting $T_i$ in a knotted spanning arc as shown in Fig. 5. However, a trivial tangle can be added to $T_i$ on the outside of the ball in Fig. 4 to create the knot $K_i$: $K_1$ is the pretzel knot $K(-2,3,7)$, and $K_2$ is the torus knot of type $(3,7)$, denoted by $T(3,7)$. Thus $K_i$ would have $T(2,5)$ as a factor of its prime decomposition. Let $\Delta_i(t)$ and $\Delta(t)$ be the Alexander polynomials of $K_i$ and $T(2,5)$ respectively, then $|\Delta_i(-1)| = 1$ must be divisible by $|\Delta(-1)| = 5$. This contradiction establishes Lemma 2.

**Remark.** $K$ is concordant to $K(-2,3,7) \neq T(3,7)$.

5. Question

Using the method of Viro [VJ], we can construct presumably non-amphicheiral knots; they are prime and the 2-fold covering spaces branched over them are symmetric. The knot of Fig. 6 is one of them. The 2-fold branched covering space is the manifold as shown in Fig. 7. Is this knot non-amphicheiral?
References


