Notes on Graph Links

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A 3-manifold $M$ is called a graph manifold if there is a union $T$ of mutually disjoint tori $T_1, \ldots, T_n$ in the interior $\overset{\circ}{M}$ of $M$ such that $M - T$ is a circle bundle over a (possibly disconnected) surface. A link $K = k_1 \cup \cdots \cup k_k$ in $S^3$ or $S^1 \times D^2$ is a graph link if the complement of a proper tubular neighborhood of $K$, called exterior of $K$, is a graph manifold.

A fundamental family of graph links is that of solid torus links in $S^1 \times D^2 < S^3$ which consist of orbits or the fixed point set of an $S^1$-action on $S^1 \times D^2$.

Given a link $K$, replacing a component of $K$ by a solid torus link in its regular neighborhood, we have a new link $K'$, which is called a solid core of $K$.

Starting from a trivial knot and repeating to take solid cores, we have an iterated solid torus link.

Recently, Soma showed that not all graph knots
are iterated (solid) torus knots by showing that
a connected sum of graph knots is again a graph knot.

This follows from the fact that an orientable regular neighborhood of a torus with an annulus attached to two parallel meridians
is a product of $S^2$ and a three-punctured sphere.

In this note, we shall characterize iterated solid
torus links among graph links by means of graphs associated
to them.

Let $M$ be a graph manifold with $I$. Note that $\partial M$ consists of tori. Let $T'$ be a parallel of $\partial M$ in $M-T$, namely, $T'=\partial V-\partial M$ for a collar neighborhood $V$ of $\partial M$ in $M-T$. We associate a graph $\mathcal{G}$ to $(M;T;\mathcal{T};T')$ as follows:

To each connected component of $M-(\mathcal{T};\mathcal{T})$ we associated a vertex $v$ and denote this component $M_v$. We color a vertex $v$ in black, if $M_v \cap \partial M=\emptyset$, and in white, otherwise. To each connected component of $\mathcal{T};\mathcal{T}$ we associate an edge $e$ and denote this component $T_e$. We define a vertex $v$ to be incident to an edge $e$, if $T_e$ is a boundary component of $M_v$. 
Thus we have a graph \( \Gamma \) for \( \Gamma \).

Note that

1. \( \Gamma \) is connected if and only if so is \( M \),
2. a white vertex of \( \Gamma \) has valency 1, and
3. a black vertex \( v \) of \( \Gamma \) has valency \( kv \) if and only if
   \( M_v \) is a circle bundle over a connected \( k_v \) punctured closed surface.

By making use of the Mayer-Vietoris and the Groenewald sequences, we have connected
Proposition 1. Suppose that \( M \) is embedded in
a 3-manifold \( W \) with \( H^1(W; \mathbb{Z}_2) = 0 \). Then
4. the graph \( \Gamma \) is a tree, and
5. for each vertex \( v \) of \( \Gamma \) of valency \( kv \),
   \( M_v \) is a circle bundle over a \( k_v \) punctured sphere.

Now let \( k \leq S^3 \) be a graph link. Let \( M \)
be its exterior with \( T \). Then we have a graph
\( \Gamma \) associated to \( (M; T \cup T') \), which satisfies the
conditions (1) \~ (5) above. In particular, each
edge \( e \) of \( \Gamma \) has distinct vertices \( v \) and \( v' \) so that
\( M_v \) and \( M_{v'} \) belong to the distinct components of \( M-T_e \).
We direct the edge $e$ as $\overrightarrow{vw}$ if $N_v$ is contained in a solid torus bounded by $T_e$. Note that if $T_e$ is unknotted, namely, $T_e$ separates $S^3$ into two solid tori, then $e$ can be directed in both directions $\overrightarrow{vw}$ and $\overrightarrow{wv}$. By the Alexander's Theorem, each edge of $\Gamma$ can be directed.

We shall say that a vertex $v$ of $\Gamma$ is minimal, if for any white vertex $v'(\neq v)$, there is a sequence of directed edges $\overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{m-1}v_m}$ such that $v_1 = v$ and $v_m = v'$.

**Theorem 1.** A graph link is an iterated solid torus link if and only if there is a graph structure of its exterior whose graph has a minimal vertex.

In particular, a graph knot is an iterated torus knot if and only if there is a graph structure of its exterior whose graph has a minimal vertex.

For the proof of this, we need to study graph links in $S^1 \times D^2$. In this case, we have also a graph associated to a graph structure of its exterior.
However, in general, not all edges can be directed. In order to specify the boundary \( S^1 \times S^1 \) of \( S^1 \times D^2 \), we color its associated vertex in red.

On the way to prove Theorem 1, we prove Theorem 2. A graph link in \( S^1 \times D^2 \) is an iterated solid torus link if and only if the red vertex of the graph associated to a graph structure of its exterior is minimal.

Note that in the graph associated to an exterior of a graph link, an edge can be "reduced", if it has two black vertices one of which has valency two. The proof of Theorems 1 and 2 is by induction on the number of vertices of valency \( \geq 3 \).

The crucial case is the case where the number of 1. In this case, the link must be a solid torus link.