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Notes on Graph Links

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A 3-manifold M is called a <u>graph manifold</u>, if there is a union T of mutually disjoint tori $T_1, \cdots, T_m$ in the interior $\mathring{M}$ of M such that $M-T$ is a circle bundle over a (possibly disconnected) surface. A link $k = k_1 \cup \cdots \cup k_r$ in S^3 or $S^1 \times D^2$ is a <u>graph link</u>, if the complement of an open tubular neighborhood of k, called exterior of k, is a graph manifold.

A fundamental family of graph links is that of <u>solid torus links</u> in $S^1 \times D^2 \subset S^3$ which consist of orbits or the fixed point set of an S^1-action on $S^1 \times D^2$.

Given a link k, replacing a component of k by a solid torus link in its regular neighborhood, we have a new link k', which is called a <u>solid cable</u> of k. Starting from a trivial knot and repeating to take solid cables, we have an <u>iterated solid torus link</u>.

Recently, Soma showed that not all graph knots are iterated (solid) torus knots by showing that a connected sum of graph knots is again a graph knot. This follows from the fact that an orientable regular neighborhood of a torus with an annulus attached to two parallel meridians is a product of S^1 and a three punctured sphere.

In this note, we shall characterize iterated solid torus links among graph links by means of graphs associated to them.

Let M be a graph manifold with T. Note that ∂M consists of tori. Let T' be a parallel of ∂M in $M-T$, namely, $T' = \partial V - \partial M$ for a collar neighborhood V of ∂M in $M-T$. We associate a graph γ to $(M; T \cup T')$ as follows:

To each connected component of $M-(T \cup T')$ we associated a vertex v and denote this component M_v. We color a vertex v in black, if $M_v \cap \partial M = \emptyset$, and in white, otherwise. To each connected component of $T \cup T'$ we associate an edge e and denote this component T_e. We define a vertex v to be incident to an edge e, if T_e is a boundary component of $\overline{M_v}$.

Thus we have a graph $\{\{v\}, \{e\}\}$.

Note that

(1) γ is connected if and only if so is M,

(2) a white vertex of γ has valency 1, and

(3) a black vertex v of γ has valency k_v if and only if M_v is a circle bundle over a connected k_v punctured closed surface.

By making use of the Meyer-Vietoris and the Gysin sequences, we have

Proposition 1. Suppose that M is connected and embedded in a 3-manifold W with $H_1(W; \mathbb{Z}_2) = 0$. Then

(4) the graph γ is a tree, and

(5) for each vertex v of γ of valency k_v, M_v is a circle bundle over a k_v punctured sphere.

Now let $k \subset S^3$ be a graph link. Let M be its exterior with T. Then we have a graph γ associated to $(M; T \cup T')$, which satisfies the conditions (1) ~ (5) above. In particular, each edge e of γ has distinct vertices v and v' so that M_v and $M_{v'}$ belong to the distinct components of $M - T_e$.

We direct the edge e as $\overrightarrow{vv'}$ if $M_{v'}$ is contained in a solid torus in S^3 bounded by T_e. Note that if T_e is unknotted, namely, T_e separates S^3 into two solid tori, then e can be directed ~~to~~ both directions $\overrightarrow{vv'}$ and $\overrightarrow{v'v}$. By the Alexander's Theorem, each edge of γ can be directed.

We shall say that a vertex v of γ is <u>minimal</u>, if for any white vertex $v'(\neq v)$, there is a sequence of directed edges $\overrightarrow{v_1 v_2}, \cdots, \overrightarrow{v_{n-1} v_n}$ such that $v_1 = v$ and $v_n = v'$.

<u>Theorem 1</u>. A graph link is an iterated solid torus link if and only if there is a graph structure of its exterior whose graph has a minimal vertex. In particular, a graph knot is an iterated torus knot if and only if there is a graph structure of its exterior whose graph has a minimal vertex.

For the proof of this, we need to study graph links in $S^1 \times D^2$. In this case, we have also a graph associated to a graph structure of its exterior.

However, in general, not all edges can be directed. In order to specify the boundary $S^1 \times S^1$ of $S^1 \times D^2$, we color its associated vertex in red.

On the way to prove Theorem 1, we prove

Theorem 2. A graph link in $S^1 \times D^2$ is an iterated solid torus link if and only if the red vertex of the graph associated to a graph structure of its exterior is minimal.

Note that in the graph associated to an exterior of a graph link, an edge can be "reduced", if it has two black vertices one of which has valency two. The proof of Theorems 1 and 2 is by induction on the number p of vertices of valency $\geqq 3$. The crucial case is the case where the number p is 1. In this case, the link must be a solid torus link.