数理解析研究所講究録
第 417 巻 1981 年 130-141

Oct. '80

ON THE BOUNDING GENUS OF HOMOLOGY 3-SPHERES

—— preliminary report ——

by Yukio Matsumoto

§0. Introduction.

By a homology 3-sphere is meant a C^∞ closed oriented 3-manifold Σ with $H_*(\Sigma;\ \mathbb{Z}) \cong H_*(S^3;\ \mathbb{Z})$. Two homology 3-spheres Σ_1, Σ_2 are said to be homology-cobordant iff there exists a C^∞ compact oriented 4-manifold W^4 with $\partial W^4 = \Sigma_1 \cup -\Sigma_2$ such that the inclusion induces an isomorphism $H_*(\Sigma_i;\ \mathbb{Z}) \longrightarrow H_*(W^4;\ \mathbb{Z})$ for each $i = i,2$. This relationship (denoted by $\Sigma_1 \sim \Sigma_2$) is an equivalence relation. $\mathcal{H}^3$ denotes the totality of the equivalence classes of homology 3-spheres. The connected sum operation makes $\mathcal{H}^3$ an abelian group.

$\mathcal{H}^3$ plays an important role in geometric topology, but nothing is known about its group structure except for the existence of an onto homomorphism $\rho\colon \mathcal{H}^3 \longrightarrow \mathbb{Z}/2$ called the Rochlin homomorphism. In fact, even the following very optimistic conjecture (VOC) remains unsettled:

(VOC) ρ is an isomorphism.

Though we have little information about $\mathcal{H}^3$, we certainly have millions of homology 3-spheres. Thus, following the philosophy in [N-R], it seems worthwhile to compile many empirical data about homology 3-spheres and 4-manifolds with homology sphere boundaries. In this note, we shall propose a genus-like invariant which seems useful for this purpose and estimate its values for certain Brieskorn homology 3-spheres.

§1. The bounding genus.

Let Σ be a homology 3-sphere. We will define the bounding genus of Σ (denoted by $|\Sigma|$) as follows: If the Rochlin invariant of Σ, $\rho(\Sigma)$, is non-zero, then define $|\Sigma|$ to be $+\infty$. If $\rho(\Sigma) = 0$, then Σ bounds a C^∞ compact spin 4-manifold W^4 whose signature satisfies $\sigma(W^4) \equiv 0 \ (16)$. By surgery we may assume that W^4 is connected and $H_1(W^4; \mathbb{Z}) = \{0\}$, in other words, that W^4 is homologically 1-connected. By taking a connected sum of W^4 and some copies of the (possibly oppositely oriented) K3 surface, we may assume $\sigma(W^4) = 0$. Now the intersection form of W^4 is an orthogonal sum of certain number, say n, of "hyperbolic planes" $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We can say briefly that if $\rho(\Sigma) = 0$, then Σ "bounds" the form $n\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The bounding genus $|\Sigma|$ is defined to be the minimum of such n.

Easy properties of $|\Sigma|$.

(1) If $\Sigma_1 \sim \Sigma_2$, then $|\Sigma_1| = |\Sigma_2|$. Thus the bounding genus (sometimes abbreviated as the b -genus) gives a function $\mathcal{H}^3 \longrightarrow \{0,1,2,3,\ldots,+\infty\}$.

(2) $|\Sigma| = 0$ iff $\Sigma = 0$ in $\mathcal{H}^3$.

(3) $|\Sigma| = |-\Sigma|$.

(4) $|\Sigma_1 + \Sigma_2| \leq |\Sigma_1| + |\Sigma_2|$, where $\Sigma_1 + \Sigma_2$ denotes the connected sum.

By these properties, the b -genus of the difference

$|\Sigma_1 - \Sigma_2|$ $(= |\Sigma_1 + (-\Sigma_2)|)$ serves as a distance function and gives a "metric" to $\mathcal{H}^3$ (but admitting infinitely distant points).

§2. The main result.

At the present stage of knowledge, we cannot find any homology 3-sphere Σ with $|\Sigma| \neq 0, +\infty$. To find such an example is equivalent to disprove (VOC). Nevertheless, we can estimate b-genera for many homology 3-spheres, and these estimates constitute our main result.

Let $\Sigma(p,q,r)$ be a Brieskorn homology 3-sphere with pairwise coprime integers $p,q,r \geqq 1$. It has a natural orientation. Let p,q,m,ℓ be integers $\geqq 1$ with $\gcd(p,q) = 1$. The main result is the following

Theorem 2.1.

(i) $|\Sigma(p, q, pqm \pm 1)| \leqq 1$ *if* m *is even*.

(ii) *If* m *is odd and* $\rho(\Sigma(p, q, pqm \pm 1)) = 0$, *then* $|\Sigma(p, q, pqm \pm 1)| \leqq (p - 1)(q - 1)/2$.

(iii) $|\Sigma(p, q, pqm - 1) + \Sigma(p, q, pq\ell + 1)| \leqq 1$ *if* $m \equiv \ell$ *mod* 2.

(iv) $|\Sigma(p, q, pqm - 1) - \Sigma(p, q, pq\ell + 1)| \leqq (p - 1)(q - 1) + 1$ *for* m, ℓ *odd*.

(v) $|\Sigma(p, q, pqm - 1) - \Sigma(p, q, pqm + 1)| \leqq (p - 1)(q - 1)$.

Comment on (ii). If m is odd, we have

$$\rho(\Sigma(p, q, pqm \pm 1)) = \rho(\Sigma(q, p, pqm \pm 1))$$

$$= \begin{cases} (1 - q^2)/8 \pmod 2), \text{ if } p \text{ is even and } q \text{ is odd}, \\ 0, \text{ if both } p \text{ and } q \text{ are odd}. \end{cases}$$

For other estimates, see §5.

To what extent are these estimates satisfactory? To content the questioner partly, we shall introduce the following concept:

<u>Definition</u>. An estimate $|\Sigma| \leqq n$ is said to be <u>hard-to-improve</u> if an improved estimate $|\Sigma| \leqq k < n$ would lead to the existence of a C^∞ closed, homologically 1-connected, spin 4-manifold M^4 with $b_2(M^4) < (11/8)\,|\sigma(M^4)|$.

Note that such a 4-manifold has been long sought after, but the existence is still unknown. As an example, put $p = 2$, $q = 3$, $m = 2$ in Theorem 2.1 (i) and take "—" sign. Then we have $|\Sigma(2,3,11)| \leqq 1$. This is hard-to-improve. To see this, note that $\Sigma(2,3,11)$ bounds a compact Milnor fiber which is a 1-connected spin 4-manifold with $b_2 = 20$, $\sigma = -16$. Thus if $\Sigma(2,3,11) = 0$ in $\mathcal{H}^3$, in other words, if $\Sigma(2,3,11)$ bounds an acyclic 4-manifold W^4, then by pasting W^4 and the Milnor fiber along their boundaries, we would obtain a "sought after" 4-manifold.

Kirby and Akbulut conjecture that $\Sigma(2,3,11) \neq 0$ [Ki_1, p.54]. In our terminology, their conjecture is interpreted as $|\Sigma(2,3,11)| = 1$.

Now we will pick up some hard-to-improve estimates from Theorem 2.1.

$|\Sigma(2,7,13)| \leqq 3$ ($p = 2$, $q = 7$, $m = 1$, -sign in (ii).)
cf. [K, p.49].

$|\Sigma(2,3,5) + \Sigma(2,3,7)| \leqq 1$ ($p = 2$, $q = 3$, $m = \ell = 1$ in (iii).)

$|\Sigma(2,3,5) - \Sigma(2,3,7)| \leqq 2$ ($p = 2$, $q = 3$, $m = 1$ in (v).)

From last two estimates follows

$$|2\Sigma(2,3,5)| \leqq 3. \qquad \text{(hard-to-improve)}$$

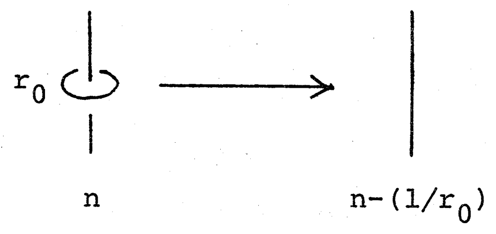
Also we can prove (by surgery)

$$|2\Sigma(2,3,7)| \leqq 1. \qquad \text{(hard-to-improve)}$$
$$|2\Sigma(2,3,5) - \Sigma(2,3,11)| \leqq 2. \qquad \text{(hard-to-improve)}$$

§ 3. Our method.

Our main tool is the Dehn-Kirby calculus [Ro] [Ki_1]. Let $L = \bigcup_{i=1}^{r} K_i$ be a smooth link in S^3, each component K_i being labelled with a "surgery coefficient" $r_i \in \mathbb{Q} \cup \{\infty\}$. Such a labelled link $(L, \{r_i\})$ is called a Dehn-Kirby diagram (briefly, a DK-diagram). Via Dehn surgery, a DK-diagram represents a C^∞ closed connected oriented 3-manifold. (A DK-presentation of a 3-manifold.) Conversely every C^∞ closed connected oriented 3-manifold has a DK-presentation, [Li]. The Dehn-Kirby calculus consists of those operations through which a DK-diagram is altered into another without changing the degree (+1)-diffeomorphism class of the corresponding 3-manifold. The following is our key lemma.

Lemma 3.1. Let K_0, K_1 be components of a DK-diagram. Suppose K_0 spans a meridian disk of K_1 whose interior does not meet other components than K_1. Suppose the coefficient r_1 of K_1 is an integer, say n. Then we can alter a part of the diagram without changing the degree (+1)-diffeomorphism type of the corresponding 3-manifold as the following picture indicates:

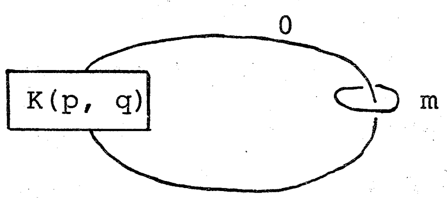


This is the process of "blowing down" the r_0-labelled component K_0. Iterative application of the inverse "blowing up" process allows one to convert all the rational coefficients into the integral ones (corresponding to the continued fraction). Several authors suggest that Rolfsen has known such a process (cf. [Ki_1, Remark 6] for example).

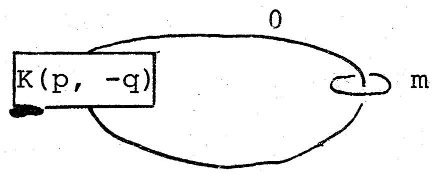
The above lemma and Seifert's argument [S] yield the following DK-presentation of the Brieskorn homology 3-spheres $\Sigma(p, q, pqm \pm 1)$:

Theorem 3.2.

$\Sigma(p, q, pqm + 1) \cong$



$\Sigma(p, q, pqm - 1) \cong$



Here K(p, q) (or K(p, -q)) denotes the (p, q)-torus knot (or its mirror image).

A DK-diagram with integral coefficients represents not only a 3-manifold but also a 4-manifold obtained by attaching

2-handles to a 4-ball along the diagram (= a framed link). Its boundary is the corresponding 3-manifold. Thus, according to Theorem 3.2, $\Sigma(p, q, pqm \pm 1)$ bounds a 1-connected 4-manifold whose intersection form is given by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix}$. If m is even, $\begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix}$ is congruent to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This proves Theorem 2.1 (i). Other estimates in the theorem are also based on Theorem 3.2, but their proofs require some additional surgery arguments.

§4. Proof of Lemma 3.1.

We use the same notation as in Lemma 3.1.

Case 1. K_0 and K_1 are unknotted. L consists of the two components K_0, K_1.

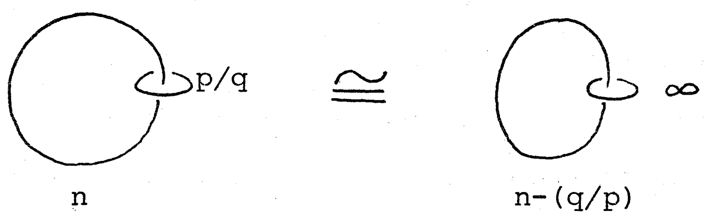
Let E be the exterior of an open tubular neighbourhood of L. Then E is identified with $S^1 \times S^1 \times [0, 1]$. The circle S^1 is parametrized by a real number mod 1. We give a coordinate (θ, φ, t) to $S^1 \times S^1 \times [0, 1]$ (where $\theta, \varphi \in \mathbb{R}/\mathbb{Z}$, $0 \leqq t \leqq 1$) so that the point $(\theta, 0, 0)$ (or the point $(0, \varphi, 0)$) traverses once the longitude ℓ_0 of K_0 (or the meridian m_0 of K_0) as θ (or φ) changes from 0 to 1, and so that the point $(\theta, 0, 1)$ (or the point $(0, \varphi, 1)$) traverses once the meridian m_1 of K_1 (or the longitude ℓ_1). Here we assume that K_0, K_1 are oriented so that the linking number is equal to 1.

Define a diffeomorphism $f: S^1 \times S^1 \times [0, 1] \longrightarrow S^1 \times S^1 \times [0, 1]$ by $f(\theta, \varphi, t) = (\theta - n\varphi, -\varphi, 1 - t)$. (Recall n is the coefficient of K_1.) f preserves the orientation of E and interchanges $S^1 \times S^1 \times \{0\}$ and $S^1 \times S^1 \times \{1\}$.

The induced homomorphism on the first homology $H_1(S^1 \times S^1 \times \{i\}; \mathbb{Z})$, $i = 0,1$, is given as follows:

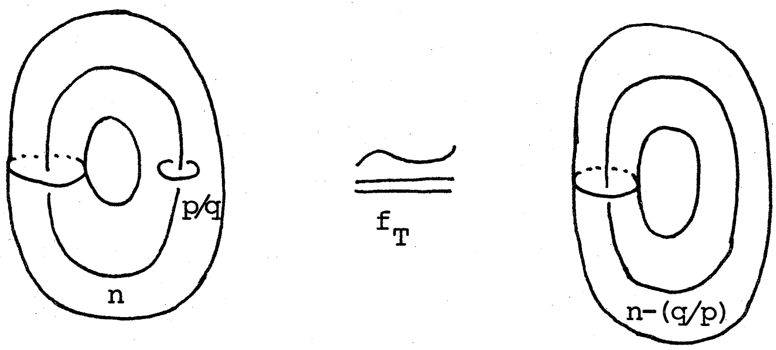
$$\begin{cases} f_*[\ell_0] = [m_1], \ f_*[m_0] = -[\ell_1] - n[m_1] \\ f_*[\ell_1] = -n[\ell_0] - [m_0], \ f_*[m_1] = [\ell_0]. \end{cases}$$

Recall that the surgery coefficient p/q attached to K_0 indicates the Dehn surgery performed along a smooth simple closed curve on $S^1 \times S^1 \times \{0\}$ whose homology class is $p[m_0] + q[\ell_0]$. Since $f_*(p[m_0]+q[\ell_0]) = (-pn+q)[m_1]+(-p)[\ell_1]$, the curve "$p/q$" on $S^1 \times S^1 \times \{0\}$ is mapped to the curve "$(-pn+q)/(-p)$" $(= n-(q/p))$ on $S^1 \times S^1 \times \{1\}$. Similarly, the curve "$n$" on $S^1 \times S^1 \times \{1\}$ is mapped to the curve "∞" on $S^1 \times S^1 \times \{0\}$. Thus f gives a degree $(+1)$-diffeomorphism between the DK-diagrams:



Since one can always get rid of a component with coefficient ∞ [Ro, p.261], the proof is done in Case 1.

Before proceeding to the general case, we note that the unknotted simple closed curve $C \equiv \{(\theta, 0, 1/2) \mid 0 \leqq \theta \leqq 1\}$ in E is a component of the fixed point set of f. Let $N(C)$ be a tubular neighbourhood of C. Then f is isotopic to the identity on $N(C)$. Now assume that $f|N(C) = id_{N(C)}$. Let $T = S^3 - \mathrm{Int}(N(C))$. T is a solid torus containing K_0, K_1 in its interior. Since the curve C spans a meridian disk of K_1, the component K_1 is the longitudinal central circle of T. Let $f_T = f|T$. Then f_T is a degree $(+1)$-diffeomorphism between the "DK-diagrams in the solid torus T" such that $f_T|\partial T = id_{\partial T}$:



Case 2. The general case. (The component K_0 is unknotted, but K_1 may not be. L may have other components than K_0, K_1.)

K_0 spans a meridian disk of K_1 whose interior intersects only the component K_1. Hence we may assume that $K_0 \subset \mathrm{Int}\, N(K_1)$, where $N(K_1)$ denotes a tubular neighbourhood of K_1. We may assume $N(K_1) \cap L = K_0 \cup K_1$. Let $V = S^3 - \mathrm{Int}\, N(K_1)$. Now identifying the solid torus T in the preceding remark with $N(K_1)$, we have the pasted diffeomorphism $f_T \cup id_V : T \cup V \longrightarrow T \cup V$ which is a degree $(+1)$-diffeomorphism between the DK-diagrams in question. This completes the proof of Lemma 3.1.

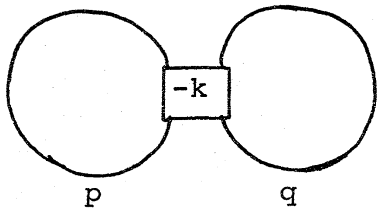
§5. Some other estimates.

Let p,q,k be integers with k odd > 1, $0 < p < q$, p, q even $\geqq 2$ and $pq = k^2 - 1$. Then we have

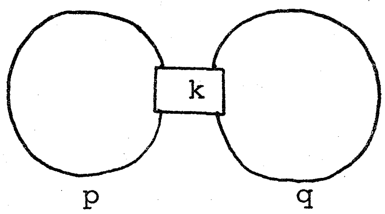
$$(5.1) \qquad |\Sigma(k,\ k \pm p,\ q \pm k)| \leqq 1.$$

This follows from the following DK-presentation:
(The same reasoning that appeared at the end of § 3 is valid.)

$\Sigma(k,\ k+p,\ k+q) \ \cong$



$\Sigma(k,\ k-p,\ q-k) \ \cong$



,

Here $\boxed{k}$ stands for k full twists.

It should be noted that the family of homology 3-spheres in (5.1) precisely coincides with the family which Casson gives in a different form ([Ki$_2$, Problem 1.37, Remarks]). Thus one may attribute the series (5.1) to him. Put $k = 2n + 1$, $p = 2n$, $q = 2n + 2$ and take $+$ sign in (5.1), then we have a one parameter sub-series:

(5.2) $$|\Sigma(2n + 1,\ 4n + 1,\ 4n + 3)| \leqq 1.$$

Recently, Noriko Maruyama obtained the following series of estimates:

(5.3) $$|\Sigma(2n + 1,\ 2n + 3,\ 4n^2 + 10n + 5)| \leqq 1.$$

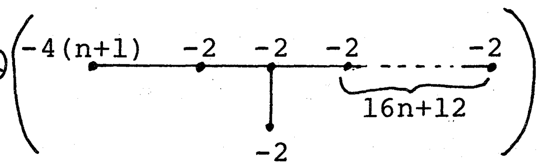
This series is interesting because it starts with $\Sigma(3,5,19)$ which is shown to be 0 in $\mathcal{H}^3$ by Fintushel-Stern [F-S].

§6. Problems and conjectures.

(6.1) Suppose we can show $|\Sigma| \leqq n$ for some homology 3-sphere Σ, then is it possible to find a 1-connected (rather than homologically 1-connected) spin 4-manifold W^4 with $\partial W^4 = \Sigma$ whose intersection form is $n\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

(6.2) Are there any non-trivial relations among Brieskorn homology 3-spheres in $\mathcal{H}^3$? For example, can we prove $2\Sigma(2,3,5) = 3\Sigma(2,3,11)$ or $\Sigma(2,3,5) + \Sigma(2,3,7) = \Sigma(2,3,11)$? (Both suspicious.)

(6.3) Conjecture: $|\Sigma(2,\ 8n+7,\ 16n+13)| \leqq 3(n+1),\ n \geqq 0.$ Note that

$\Sigma(2,8n+7,16n+13) = \partial$ 

, cf. [N-R].

By Theorem 2.1 (ii), we can show

$$|\Sigma(2,\ 8n+7,\ 16n+13)| \leqq 4n+3.$$

(6.4) Conjecture: (improving Theorem 2.1 (ii)) If m is odd and $\rho(\Sigma(p,\ q,\ pqm \pm 1)) = 0$, then

$$|\Sigma(p,\ q,\ pqm+1)| \leqq (p-1)(q-1)/2 - 1.$$

(6.5) It is shown by Theorem 2.1 (iii) (v) that $|2\Sigma(p,\ q,\ pqm-1)| \leqq (p-1)(q-1)+1$. Is it possible to prove $|2\Sigma(p,\ q,\ pqm+1)| \leqq (p-1)(q-1)-1$? (True for $p=2,\ q=3,\ m=1$.)

References

[F-S] R. Fintushel and R. Stern, An exotic free involution on S^4, Topology Festival at Cornell University, Ithaca, 1980.

[Ki_1] R. Kirby, A calculus for framed links in S^3, Inventiones math., 45 (1978), 35-56.

[Ki_2] R. Kirby, Problems in low dimensional manifold theory, Proc. Symp. Pure Math. 32 (1978).

[Li] W.B.R. Lickorish, A representation of orientable combinatorial 3-manifolds, Ann. of Math. 76 (1962), 531-540.

[N-R] W.D. Neumann and F. Raymond, Seifert manifolds, plumbing, μ-invariant and orientation reversing maps, Lecture Notes in Math. #664 (1977), 163-196, Springer.

[Ro] D. Rolfsen, Knots and links, Mathematics Lecture Series, Publish or Perish, Inc. Berkeley (1976).

[S] H. Seifert, Topologie dreidimensionaler gefaserter Raume, Acta Mathematica 60 (1933), 147-288. English translation: Topology of 3-dimensional fibered spaces, in Seifert-Threlfall, A textbook of Topology Academic Press, New York (1980).

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo