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# J-groups of lens spaces

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§1.  Introduction

The standard lens space mod m is the orbit manifold

$$L^n(m) = S^{2n+1}/Z_m \quad (Z_m = \{z \in S^1 : z^m = 1\})$$

of the (2n+1)-sphere $S^{2n+1}(\subset C^{n+1})$ by the diagonal action

$$z(z_0, \cdots, z_n) = (zz_0, \cdots, zz_n).$$

The J-groups of lens spaces were studied by several authors (e.g. [2],[5],[6],[8],[10] and [11]).

Let $\eta_m$ be the canonical complex line bundle over $L^n(m)$. Then we have the following theorem by making use of the results due to J.F.Adams [1] and D.Quillen [12].

<u>Theorem 1.1.</u>  Let p be a prime and let $r(\eta_{p^r}^i - 1) \in \widetilde{KO}(L^n(p^r))$ $(p^r \geq 3)$ be the real restriction of the stable class of the i-fold tensor product of $\eta_{p^r}$. Then the order of the J-image

$$Jr(\eta_{p^r}^i - 1) \in \tilde{J}(L^n(p^r))$$

is equal to $p^{f_p(n,r;\nu)}$,

$$f_p(n,r;\nu) = \max\{s - \nu + [n/p^s(p-1)]p^{s-\nu} : \nu \leq s < r \text{ and } p^s(p-1) \leq n\},$$

where $\nu = \nu_p(i)$ is the exponent of p in the prime power

decomposition of i and max $\phi = 0$.

In this lecture, we prove the above theorem only the case

p=2, since we can prove the above theorem for odd prime p in the

similar way (see [10]).

Remark 1.2.   By Theorem 1.1 and Proposition 1.3 below, we

can determine the order of $Jr(\eta_m^i - 1)$ in $\tilde{J}(L^n(m))$ for any m.

Let $L_0^n(m)$ be the 2n-skeleton of $L^n(m)$ and $m = \Pi p^{r(p)}$ be the

prime power decomposition of m.   Then we have

Proposition 1.3.   (i) The sequence

$$0 \to \tilde{J}(S^{2n+1}) \to \tilde{J}(L^n(m)) \xrightarrow{i^*} \tilde{J}(L_0^n(m)) \to 0$$

is a split extension for odd m.

(ii) There exists a natural isomorphism

$$f = \oplus \pi_p^* : \tilde{J}(L_0^n(m)) \to \oplus_p \tilde{J}(L_0^n(p^{r(p)})) \quad (m:odd),$$

$$f = \oplus_p (i_p \circ \pi_p)^* \oplus \pi_2^* : \tilde{J}(L^n(m)) \to \oplus_{p:odd\ prime} \tilde{J}(L_0^n(p^{r(p)})) \oplus \tilde{J}(L^n(2^{r(2)})) \quad (m:even),$$

which satisfies

$$f(Jr(\eta_m^i - 1)) = \sum_p Jr(\eta_{p^{r(p)}}^i - 1),$$

where $\pi_q : L^n(q) \to L^n(m)$ and $i_p : L_0^n(p^{r(p)}) \to L^n(p^{r(p)})$ are the natural

projection and inclusion, respectively.

Proof.   (i) is immediate from Puppe exact sequence in KO-

theory and the fact that $\tilde{J}(L_0^n(m))$ is of odd order if m is odd.

(ii) We can show the similar result for $\widetilde{KO}$ instead of $\tilde{J}$ by

noticing that f is surjective and the both sides groups have the

same order (cf.   [3, Lemma 2.3 (ii)] and [13. Th.(0.1)]).   The

last equality follows from the definitions of $f$, $\eta_m$ and $\eta_{p^{r(p)}}$.

$$\text{q.e.d.}$$

§2.  The structure of $\tilde{J}(L^n(2^r))$

Let $\eta$ be the canonical complex line bundle over $L^n(2^r)$ and $\rho$ be the non-trivial real line bundle over $L^n(2^r)$ and put

$$\sigma(s) = \eta^{2^s} - 1, \quad \sigma(0) = \sigma \quad \epsilon \tilde{K}(L^n(2^r)),$$

$$\kappa = \rho - 1 \quad \epsilon \widetilde{KO}(L^n(2^r)).$$

Then $\widetilde{KO}(L^n(2^r))$ is generated additively by the elements

$$\kappa \quad \text{and} \quad r(\sigma^d \sigma(s)) \quad (0 \le s \le r-2, \ 0 \le d < 2^s),$$

and its explicit additive and multiplicative structures are known ([9, Th.1.9]).

The calculation of Adams operations $\Psi^k$ on $\tilde{K}(L^n(2^r))$ and the property $r \circ \Psi_c^k = \Psi_R^k \circ r$ of Adams operations on $\tilde{K}$ and $\widetilde{KO}$ imply the following

Lemma 2.1.  Let $J : \widetilde{KO}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ be the J-homomorphism. Then Ker $J$ is generated additively by the elements

$$r(\sigma^d(1+\sigma)\sigma(s)) \quad (0 \le s \le r-1, \ 0 \le d < 2^s - 1).$$

From now on, we use the following notation

$$\alpha_s = Jr\sigma(s) \quad \epsilon \tilde{J}(L^n(2^r)).$$

Here, we notice that $\alpha_s = 0$ if $s \ge r$ and $\alpha_{r-1} = 2J\kappa$, since $\eta^{2^s} = 1$ if $s \ge r$ and $\eta^{2^{r-1}} = 2\rho$.

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From the above lemma, we see easily the following

Proposition 2.2. $\tilde{J}(L^n(2^r))$ is generated by

$$J\kappa \quad \text{and} \quad \alpha_s \quad (0 \leq s \leq r-2).$$

Combining the relations of $\widetilde{KO}(L^n(2^r))$ given in [9, Th.1.9] and the relations arisen from Ker J in Lemma 2.1, we have the following theorem on the group structure of the reduced J-group $\tilde{J}(L^n(2^r))$ $(r \geq 2)$, where

$$a_s = [n/2^s], \quad b_s = n-2^s a_s \quad (0 \leq s < r),$$

$$X(d,v) = \Sigma_{j \in Z}(-1)^{j(2^v+1)}\binom{2d}{d+2^v j},$$

$$Y(d,v) = \Sigma_{j \in Z}\binom{2d-1}{d+2^v(2j+1)}.$$

Theorem 2.3. (i) [5, Th.4.5] $J:\widetilde{KO}(L^n(4)) \cong \tilde{J}(L^n(4))$.

(ii) The relations of $\tilde{J}(L^n(2^r))$ for $r \geq 3$ are given as follows:

(a) The case $n \not\equiv 1 \bmod 4$:

(2.3.1) $2^{1+a_{r-1}}J\kappa = 0$, $2^{r-1+2a_1}\alpha_0 = 0$, $2^{r-1-s+a_s}\alpha_s = 0$ $(1 \leq s \leq r-2)$.

(2.3.2) $2^{a_{r-1}}J\kappa + \Sigma_{v=0}^{r-2}2^{r-1-v(1+a_{r-1})-2}\alpha_v = 0$ if $a_1 \geq 2^{r-2}$.

(2.3.3) $2^{r-s-2+a_s}\alpha_s + \Sigma_{v=0}^{s-1}2^{r-s-3+2^{s-v}(1+a_s)}\alpha_v = 0$ $(1 \leq s \leq r-2, \ 2^s \leq a_1)$.

(2.3.4) $\Sigma_{v=0}^{s}(-1)^{2^{s-v}}2^{r-s-4+2^{s+1-v}(a_{s+1}+\delta)}X(d,v)\alpha_v = 0$

$$(1 \leq s \leq r-2, \ 1 \leq d < 2^s, \ 2^s+d \leq a_1),$$

where $\delta = 1$ if $2d \leq b_{s+1}$, $= 0$ otherwise.

(2.3.5) $2^{2i-2}\alpha_0 - \Sigma_{v=1}^{t}Y(i,v)\alpha_v = 0$ where $2^t \leq i < 2^{t+1}(a_1 < i < 2^{r-1})$.

(b) The case $n \equiv 1 \bmod 4$: The relations in (a), excluded the

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one in (2.3.4) for s=r-2, $2d=1+b_{r-1}$ and the one in (2.3.5) for

$i=a_1+1$, and in addition,

(2.3.6) $2^{2i-2}\alpha_0 - \Sigma_{v=1}^{t} Y(i,v)\alpha_v = 0$ where $2^t \leq a_1+1 < 2^{t+1}$ if $a_1 < 2^{r-2}$.

For the special case that $n=2^{r-1}a$, we can reduce the

relations of $\tilde{J}(L^n(2^r))$ in (ii) of the above theorem to more simple

ones, and $\tilde{J}(L^n(2^r))$ is given by the following explicit form,

where $Z_h(x)$ denotes the cyclic group of order h generated by the

element x.

Theorem 2.4. If $n=2^{r-1}a$ ($r \geq 3$, $a \geq 2$), then $\tilde{J}(L^n(2^r))$ is the

direct sum

$$Z_{2^{r-1}-n}{}^{<\alpha_0>} \oplus \oplus_{s=1}^{r-2} Z_{2^{a_s-1}}{}^{<\alpha_s-2^{a_s-1-a_s+1}\alpha_{s-1}>}$$

$$\oplus Z_{2^{a_{r-1}}}{}^{<J\kappa+2^{a_{r-2}-a_{r-1}}\alpha_{r-2}>}.$$

By using the above theorem, the known fact about the kernel

of $i^*:\widetilde{KO}(L^n(2^r)) \to \widetilde{KO}(L^{n-1}(2^r))$ ([9, Prop.4.4]) and the calculation

of Adams operation $\psi^3$ on $\widetilde{KO}(L^n(2^r))$, we can determine the kernel of

(2.5) $i^* : \tilde{J}(L^n(2^r)) \to \tilde{J}(L^{n-1}(2^r))$

as follows :

Proposition 2.6. $i^*$ in (2.5) is isomorphic if $n \equiv 3$ mod 4,

epimorphic otherwise, and

$$\text{Ker } i^* = \begin{cases} Z_4{}^{<2J\bar{\sigma}^{-2m+1}>} & \text{if } n = 4m+2 \\ Z_2{}^{<J\bar{\sigma}^{-2m+1}>} & \text{if } n = 4m+1 \\ Z_u{}^{<J\bar{\sigma}^{-2m}>} & \text{if } n = 4m>0, \end{cases}$$

where $\bar{\sigma}=r(\eta-1) \in \widetilde{KO}(L^n(2^r))$ and

$$u = 2^{\min\{r+1,\ell+2\}} \text{ for } n=4m=2^\ell q \text{ with } (2,q)=1.$$

## §3. Proof of Theorem 1.1

To prove Theorem 1.1 for $p=2$, we prepare some lemmas.

<u>Lemma</u> 3.1. The following equality holds in $\tilde{J}(L^n(2^r))$ $(r\geq 2)$:

$$Jr(\eta^i-1) = Jr\sigma(\nu) = \alpha_\nu \quad \text{for} \quad i\geq 1,$$

where $\nu=\nu_2(i)$ is the exponent of 2 in the prime power decomposition of i.

<u>Proof.</u> By Lemma 2.1, we notice that the kernel of $J$: $\widetilde{KO}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ is generated additively by

$$r(\eta^j\sigma(s)) \quad (0\leq s<r, \ 1\leq j<2^s).$$

If $2^s\leq i<2^{s+1}$, then $\eta^i-1 = \eta^i\sigma(s)+\eta^j-1$ wher $j = i-2^s$. If $j>0$ in addition, then $Jr(\eta^i-1) = Jr(\eta^j-1)$ by the above notice and $\sigma(s)=0$ $(s\geq r)$. By continuing this process, we have the desired equality.

q.e.d.

Now, let $f_2(n,r;\nu)$ be the non-negative integer such that

$$\#Jr\sigma(\nu) = \#\alpha_\nu = 2^{f_2(n,r;\nu)} \quad \text{in} \quad \tilde{J}(L^n(2^r)) \quad (n\geq 0, \ r\geq 2),$$

where $\#\alpha$ denotes the order of $\alpha$. Then by the definition of $\alpha_\nu$, we see that

(3.2) $f_2(n,r;\nu) = 0$ if $n = 0$ or $\nu \geq r$.

<u>Lemma</u> 3.3. If $n=2^{r-1}a$ and $r\geq 3$, then

$$f_2(n,r;\nu) = r-1-\nu+2^{r-1-\nu}a \quad \text{for} \quad n>0, \ 0\leq\nu<r.$$

<u>Proof.</u> The lemma for $a\geq 2$ is easily seen from Theorem 2.4 and $\alpha_{r-1}=2J\kappa$.

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Consider the case $n = 2^{r-1}$. Then, by Proposition 2.6,

$$\# J\bar{\sigma}^{-2m} = 2^{r+1} \quad \text{in} \quad \tilde{J}(L^{2^{r-1}}(2^r)) \quad (4m = 2^{r-1}).$$

On the other hand, $2^r \bar{\sigma}^{-2m} = 2^{r+4m-2}\bar{\sigma}$ in $\widetilde{KO}(L^{2^{r-1}}(2^r))$ by [7, Lemma 2.3]. Thus, we obtain

$$(3.4) \qquad \#\alpha_0 = \#J\bar{\sigma} = 2^{r-1+2^{r-1}}.$$

Furthermore, we have the following relations in $\tilde{J}(L^{2^{r-1}}(2^r))$ by Theorem 2.3 :

$$(3.5) \qquad \begin{aligned} 2^{a_v}\alpha_v &= 2^{a_{v-1}+1}\alpha_{v-1} \quad (1 \le v \le r-3), \\ 2^2\alpha_{r-2} + 2^5\alpha_{r-3} &= 0 = 2J\kappa + 2^2\alpha_{r-2}. \end{aligned}$$

The relations (3.4) and (3.5) imply immediately

$$\#\alpha_v = r-1-v+2^{r-1-v} \quad (0 \le v < r),$$

which is the equality for $a = 1$. \hfill q.e.d.

Consider the commutative diagram $(r \ge 3)$

$$(3.6) \qquad \begin{array}{ccc} \mathrm{Ker}\, i^* \subset \tilde{J}(L^n(2^r)) & \xrightarrow{\ i^*\ } & \tilde{J}(L^{n-1}(2^r)) \\ \pi^* \downarrow & & \downarrow \pi'^* \\ \mathrm{Ker}\, i'^* \subset \tilde{J}(L^n(2^{r-1})) & \xrightarrow{\ i'^*\ } & \tilde{J}(L^{n-1}(2^{r-1})) \end{array}$$

of the induced homomorphisms, where $i$ and $i'$ are the inclusions and $\pi$ and $\pi'$ are the natural projections. Then we have the following

Lemma 3.7. If $n \not\equiv 0 \bmod 2^{r-1}$ $(r \ge 3)$, then

$$\pi^* | \mathrm{Ker}\, i^* : \mathrm{Ker}\, i^* \longrightarrow \mathrm{Ker}\, i'^*$$

is isomorphic.

Proof. If $n = 4m = 2^\ell q$ (q:odd), then the assumption $n \not\equiv 0 \bmod 2^{r-1}$

implies $r-1>\ell$ and so $\min\{r+1,\ \ell+2\} = \ell+2 = \min\{r,\ \ell+2\}$. Thus, we see immediately the lemma by Proposition 2.6, by noticing that $\pi^*r\eta = r\pi^*\eta = r\eta$ and hence $\pi^*J\bar\sigma^{-i} = J\bar\sigma^{-i}$.                q.e.d.

**Lemma 3.8.** If $n\not\equiv 0 \bmod 2^{r-1}$ $(r\geqq 3)$, then

$$f_2(n,r;\nu) = \max\{f_2(n-1,r;\nu),\ f_2(n,r-1;\nu)\}.$$

**Proof.** Consider the diagram (3.6). Then the definition of $f_2(n,r;\nu)$ implies that

$$f_2(n,r;\nu) \geqq \max\{f_2(n-1,r;\nu),\ f_2(n,r-1;\nu)\},$$

since $i^*\alpha_\nu = \alpha_\nu$ and $\pi^*\alpha_\nu = \alpha_\nu$. Moreover, if $f_2(n,r;\nu) > \max\{f_2(n-1,r;\nu),f_2(n,r-1;\nu)\}$, then the non-zero element $2^{f_2(n,r;\nu)-1}\tilde\alpha_\nu$ in $\tilde J(L^n(2^r))$ is mapped to $0$ by $i^*$ and $\pi^*$. This contradicts Lemma 3.7. Thus we have the lemma.                q.e.d.

**Proof of Theorem 1.2.** By (3.2), it is sufficient to show that

(3.9)   $f_2(n,r;\nu) = \max\{s-\nu+[n/2^s]2^{s-\nu} : \nu\leqq s<r \text{ and } 2^s\leqq n\}$  $(0\leqq\nu<r)$.

(3.9) for $r=2$ is easy consequence of Theorem 2.3 (i) and [4, Th.B]. By Lemma 3.3, (3.9) holds if $r\geqq 3$ and $n\equiv 0 \bmod 2^{r-1}$.

For the case $r\geqq 3$ and $2^{r-1}a<n<2^{r-1}(a+1)$, assume inductively that (3.9) holds for $(n-1,r;\nu)$ and $(n,r-1;\nu)$ instead of $(n,r;\nu)$. Then, we see easily that the right hand side of the equality in Lemma 3.8 is equal to

$$\begin{cases} f_2(n,r-1;\nu) & \text{if } a=0, \\ \max\{f_2(n,r-1;\nu),\ r-1-\nu+[(n-1)/2^{r-1}]2^{r-1-\nu}\} & \text{if } a>0, \end{cases}$$

and hence to the right hand side of (3.9). Thus Lemma 3.8 implies (3.9) by the induction on n and r.

These complete the proof of Theorem 1.2.                    q.e.d.

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