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On the β-family in stable homotopy of spheres at the prime 3

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The stable homotopy of spheres $\pi^s_{*(p)}$ localized at an odd prime p has the Adams filtration associated to BP, the Brown-Peterson spectrum at p:

$$\pi^s_{*(p)} \supset F^1 \supset F^2 \supset \ldots,$$

and F^1/F^2 is a direct summand isomorphic to the image of J. In case $p \geqq 5$, there is an infinite family $\{\beta_t\}$, called the β-family, in $F^2 - F^3$ (Smith,[10]), but in case $p = 3$ only a part of β's exists, namely, β_1, β_2, β_3, exist (Toda,[12]), β_4 does not and β_5 does (Oka,[5]), β_6 does (Nakamura,[4], Tangora[11]), β_7 and β_8 do not (Ravenel, unpublished), and β_9 does exists (Ravenel, Knapp[2],[3]). Karl Heinz Knapp [2],[3] proved that for $p \geqq 5$, β_{p+1} is not in the image of the bi-stable J-homomorphism $J' : \pi^s_*(SO)_{(p)} \longrightarrow \pi^s_{*(p)}$. This gives a counterexample to the conjecture of G.W. Whitehead : J' is onto. Unfortunately his proof does not work for $p = 3$, because β_4 does not exist. He told me the first candidate giving a counterexample at p=3 is β_{10} , and asked me whether or not β_{10} exists. In 1977-1978, Doug Ravenel wrote me that the BP_*-module $BP_*/(3, v_1^2, v_2^9)$ is realized by an 8-cell complex and it would follow that β_t exists whenever $t \not\equiv 4,7$ or $8 \mod 9$ (cf. [8],p.144). His proof of the realization is based on his extensive calculation of BP-Adams spectral sequence up to dim $\leqq 144$, and I do not know his publication of the result. I have, however, proved that the realization of $BP_*/(3, v_1^2, v_2^9)$ implies the existence of β_t for $t \equiv 0,1,2,5,6 \mod 9$, and I feel there is a gap in proving for $t \equiv 3 \mod 9$. My proof on β_{10} here is independent of Ravenel's. I use the result on $\pi^s_{*(3)}$ up to dim 80, though dim $\beta_{10} = 154$.

Lemma 1. For $t = 1,5$, there is a map

$$b_t : S^{16t} \longrightarrow V = S^0 \cup_3 e^1 \cup_{\alpha_1} e^5_\alpha \cup_3 e^6$$

such that $(b_t)_* = v_2^t$ and $\pi_0 b_t = \beta_t \in \pi_*^S$, where $\pi_0 : V \text{—} S^6$ collapses the 5-skeleton of V.

<u>Proof</u>. $V = V(1)$ in [10],[15], and b_1 is the attaching map of the top cell in $V(1\frac{1}{4}) = V \cup e^{17}$. By the results on π_*^S , dim, ≤ 80 , β_5 has a factorization $S^{80} \xrightarrow{b_5} V \longrightarrow S^6$. Then b_5 has a property $(b_5)_* = v_2^5$ by the Geometric Boundary Theorem (=G.B. Th.) [1].

Put $M = S^0 \cup_3 e^1$. Then V is a mapping cone of some map $\alpha: \Sigma^4 M \longrightarrow M$, and we have the cofibrations.

$$M \xrightarrow{i_1} V \xrightarrow{\pi_1} \Sigma^5 M, \qquad S^0 \xrightarrow{i} M \xrightarrow{\pi} S^1.$$

Using a similar homotopy as in [9], p.374, we can construct a map $u: V \wedge V \to \Sigma^5 M$ such that $u(i_1 \wedge 1) = \mu_M(1 \wedge \pi_1)$, $u(1 \wedge i_1) = \mu_M(\pi_1 \wedge 1)$, where $\mu_M = M \wedge M \to M$ is the multiplication. u is unique, and $\pi u : V \wedge V \to S^6$ gives the self-duality of V. If V has a multiplication, u is defined to be $V \wedge V \xrightarrow{\mu} V \longrightarrow V/M = \Sigma^5 M$, but μ does not exist for $p = 3$ (Toda). Let U be the fiber of u, then there is a commutative diagram :

$$\begin{array}{ccccccc}
\Sigma^4 M & \longrightarrow & U & \longrightarrow & V \wedge V & \xrightarrow{u} & \Sigma^5 M \\
\downarrow \iota \wedge 1 & & \downarrow \gamma & & \downarrow q & & \downarrow \iota \wedge 1 \\
BP \wedge \Sigma^4 M & \xrightarrow{1 \wedge \alpha} & BP \wedge M & \longrightarrow & BP \wedge V & \longrightarrow & BP \wedge \Sigma^5 M
\end{array}$$

Here $\iota : S^0 \to BP$, $i_0 = i_1 i : S^0 \to V$ are the inclusion and q is a unique map such that $q(i_0 \wedge 1) = \iota \wedge 1 = q(1 \wedge i_0)$ [5]. Then we have a comm. diagram

$$\begin{array}{ccccccccc}
0 & \to & BP_*(\Sigma^4 M) & \to & BP_*(U) & \to & BP_*(V \wedge V) & \to & 0 \\
 & & \| & & \downarrow \gamma_\# & & \downarrow q_\# & & \\
0 & \to & BP^*(\Sigma^4 M) & \xrightarrow{v_1} & BP\ (M) & \longrightarrow & BP_*(V) & \longrightarrow & 0,
\end{array}$$

where $q_\#$ is induced by $BP \wedge V \wedge V \xrightarrow{1 \wedge q} BP \wedge BP \wedge V \xrightarrow{u \wedge 1} BP \wedge V$ and $\gamma_\#$ is similarly defined. The composite $BP_*(V) \xrightarrow{(i_0 \wedge 1)_*} BP_*(V \wedge V) \xrightarrow{q_\#} BP_*(V)$ is the identity and $(i_0 \wedge 1)_*$ is isomorphic in dim $\equiv 0 \mod 4$.

Hence $q_\#$ is the inverse of the BP_*BP-comodule homomorphism $(i_0\wedge 1)_*$ or the zero homomorphism. Therefore $q_\#$ is a comodule homomorphism, though it is not a induced homomorphism of BP-homology. Similarly, $\gamma_\#$ is also a comodule homomorphism.

Now $(b_5\wedge b_5)_*(1)\in BP_{160}(V\wedge V)$ and we have $q_\#(b_5\wedge b_5)_*(1) = v_2^5v_2^5 = v_2^{10}$ because q gives the multiplication on $BP\wedge V$ such that $1\wedge i_0 : BP \longrightarrow BP\wedge V$ is a map of ring spectra [16]. By the G.B.th., we have

Theorem 1. The composite $S^{160} \xrightarrow{b_5\wedge b_5} V\wedge V \xrightarrow{u} \Sigma^5 M \xrightarrow{\pi} S^6$ projects to $\beta_{10}\in \mathrm{Ext}^{2,*}(BP_*, BP_*)$. Thus $\beta_{10}\in \pi^s_{154}$ exists and its order is 3.

Remark. Let D be the Spanier-Whitehead dual functor (contravariant). Here the duality map for a finite CW complex (spectrum) X is taken to be the map $X\wedge D(X) \longrightarrow S^0$. Then $D(V) = \Sigma^{-6}V$, $D(S^n) = S^{-n}$ so $D(b_5) : \Sigma^{-6}V \longrightarrow S^{-80}$. Then the above $\beta_{10} = \pi u(b_5\wedge b_5)$ is the composite $D(b_5)b_5$. As in Lemma 1, β_1 has a similar property, so we have also

$$\beta_2 = \pi u(b_1\wedge b_1), \quad \beta_6 = \pi u(b_1\wedge b_5) \text{ in } \pi^s_*(3).$$

Let $V' = S^0u_3e^1u_{\alpha 2}e^9u_3e^{10}$, $V'' = S^0u_3e^1u_{\alpha 3}e^{13}u_3e^{14}$, then

$BP_*V' = BP_*/(3, v_1^2)$, $BP_*V'' = BP_*/(3, v_1^3)$. Let $\lambda : \Sigma^4V \longrightarrow V'$, $\lambda' : \Sigma^4V' \longrightarrow V''$ be the maps such that λ_* and λ'_* are the multiplication by v_1.

Let $\bar\beta : \Sigma^{16}V\wedge B \longrightarrow V\wedge B$ be the map in [7], and define $B_t : \Sigma^{16t-11} B \longrightarrow V$ to be the composite

$$\Sigma^{16t-11} B \xrightarrow{i_0\wedge 1} \Sigma^{16t-11} V\wedge B \xrightarrow{\bar\beta^t} \Sigma^{-11} V\wedge B \xrightarrow{1\wedge k} V.$$

Then $BP_*(B) = BP_* + \Sigma^{11}BP_*$, and $(B_t)_* = 0$ on the bottom cell generator and $(B_t)_* = v_2$ on the top cell generator.

Lemma 2. For $s = 0,1,2,5,6$, there is a map $c_s: S^{16s+4} \longrightarrow V'$ such that $(c_s)_* = v_1v_2^s$.

Proof. For $s = 0$, put $c_0 = \lambda i_0$, and for $s = 1,5$, put $c_s = b_s\lambda$. Let $j : S^0 \to B$ be the inclusion. Then, for $s = 2,6$, $B_s j = i_1\xi_s$ for some $\xi_s \in \pi_{16s-11}(M)$, and $\lambda B_s j = i_1'\alpha\xi_s = 0$ because $\alpha o \pi_{16s-11}(M) \subset \pi_{16s-7}(M) = 0$. Hence $\lambda B_s = c_s k$ and $(c_s)_* = v_1 v_2^s$. For $s = 3$, $B_3 j = i_1 i\varepsilon'$, so $\lambda B_3 j = i_1\alpha i\varepsilon' \neq 0$ because $\alpha_1\varepsilon' = \beta_1^4 \neq 0$ [14]. From this, the lemma is not true for $s = 3$. We have $\lambda'\lambda \; B_3 j = 0$, and

Lemma $2'$. There is a map $c_3' : S^{56} \to V''$ with $(c_3')_* = v_1^2 v_2^3$.

Now the map $v_2^3 : BP_* \to BP_*/(3,v_1^2)$ is the elment in $H^0 BP_*/(3,v_1^2) = H^0 BP_*(V')$, and $d_5(v_2^3) \neq 0$ in the BP-Adams spectral sequence converging to $\pi_*(V')$. V' has a multiplication [6], so the spectral sequence is multiplicative. Althouth the multiplication on V' is not associative (because, the sub ring spectrum M is not associative [13],[15]), we have $d_5(x^3) = 3x^2 d_5(x)$ for $x = v_2^3$, so $d_5(v_2^9) = 0$. The next differentials possibly killing v_2^9 are d_9 , d_{13} ,... . By calculating $H^* BP_*/(3,v_1^2)$ up to dim $\leqq 144$, Ravenel claimed that there are no such differentials, that is,

Claim. $v_2^9 \; H^{0,144} BP_*(V')$ is a permanent cycle.

Then there is a map $v : S^{144} \to V'$ with $v_* = v_2^9$. The composite $\tilde{v} : \Sigma^{144} V' \xrightarrow{v\wedge 1} V' \wedge V' \to V'$ also satisfies $(\tilde{v})_* = v_2^9$, and hence the mapping cone of $\tilde{v}$ clearly realizes $BP_*/(3,v_1^2,v_2^9)$.

Theorem 2. Claim implies that β_t is a permanent cycle if $t \equiv 0, 1, 2, 5, 6 \bmod 9$.

Proof. Put $t = 9k+s$, $0 \leq s < 9$. The composite

$$c_t : S^{16t+4} = S^{144k+16s+4} \xrightarrow{c_s} \Sigma^{144k} V' \xrightarrow{\tilde{v}^k} V'$$

satisfies $(c_t)_* = v_1 v_2^t$. Then , using the G.B.Th. twice, we see that $\beta_t \in H^2 BP_*$ is a permanent cycle and converges to

$\pi_0' c_t : S^{16t+4} \longrightarrow V' \longrightarrow S^{10}$.

Using Lemma 2' instead of 2, we have

<u>Theorem</u> 2'. If $v_2^9 \quad H^{0,144}BP_*/(3,v_1^3) = H^{0,144}BP_*(V'')$ is a permanent cycle, then β_{9k+3} is a permanent cycle.

<u>Theorem</u> 2". If claim holds and the corresponding homotopy element v satisfies $\{v, 3, \beta_1^4\} = \{0\}$, then β_{9k+3} is a permanent cycle.

For, the additional assumption implies the existence of the map c_{12} as in Lemma 2.

References

[1] D. Johnson, H. Miller, W. Wilson and R. Zahler, Boundary homomorphisms in the generalized Adams spectral sequence and the non-triviality of infinitely many γ_t in stable homotopy, Reunion sobre teoria de homotopia, Northwestern Univ.1974, Soc. Mat. Mexicana, 1975, 47-59.

[2] K.H. Knapp, Some applications of K-theory to framed bordism : e-invariant and transfer, Habilitationsschrift, Univ. Bonn, 1979.

[3] K.H. Knapp, On the bi-stable J-homomorphism, preprint.

[4] O. Nakamura, Some differentials in the mod 3 Adams spectral sequence, Bull. Sci. Engrg. Div. Univ. Ryukyus (Math. Nat. Sci.), 19(1975), 1-26.

[5] S. Oka, The stable homotopy groups of spheres II, Hiroshima Math. J. 2(1972), 99-161.

[6] S. Oka, Ring spectra with few cells, Japan. J. Math. 5(1979), 81-100.

[7] S. Oka and H. Toda, 3-Primary β-family in stable homotopy, Hiroshima Math.J. 5(1975), 447-460.

[8] D. Ravenel, A novice's guide to the Adams-Novikov spectral sequence, Springer Lecture Notes No. 658, 1978.

[9] E.H. Spanier, Function spaces and duality, Ann. of. Math. 70 (1959), 338-378.

[10] L. Smith, On realizing complex bordism modules, Amer. J. Math. 92 (1970), 739-856.

[11] M.C. Tangora, Some homotopy groups mod 3, Reunion sobre teoria de homotopia, Northwestern Univ. 1974, Soc. Mat. Mexicana, 1975, 227-245.

[12] H. Toda, p-Primary components of homotopy groups IV , Mem. Coll. Sci. Univ. Kyoto Ser. A, 32(1959), 288-332.

[13] H. Toda, Extended power of complexes and applications to homotopy theory, Proc. Japan Acad. Sci. 44(1968), 198-203.

[14] H. Toda, On iterated suspensions III, J. Math. Kyoto Univ. 8(1968), 101-130.

[15] H. Toda, On spectra realizing exterior parts of the Steenrod algebra. Topoloty 10 (1971), 53-66.

[16] S. Yanagida and Z. Yosimura, Ring spectra with coefficients in V(1) and V(2),I, Japan. J. Math. 3(1977), 191-222.