

Theory of Hypermatroids

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Summary

Hypermatroid is a generalized notion of matroid and network flow. However, it is not a mere generalization but it gives us a deep insight into matroids and network flows and also provides us with new significant problems which are overlooked. This is an abstract of the series of the author's papers [1] ~ [7].

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If E is a finite set, we let $\mathfrak{S}_0(E)$ denote the linear space of real-valued *modular functions* on 2^E which corresponds closely to the vector space R^E except on ϕ . Let μ^ϕ denote the *constant function* in $\mathfrak{S}_0(E)$ such that $\mu^\phi(X) = 1$ for any $X \subseteq E$ and let μ^i denote the *unit function* such that $\mu^i(X) = 1$ if $i \in X$ and 0 otherwise. For any $\xi \in \mathfrak{S}_0(E)$, we define

$$\text{car}^\pm \xi = \{i \mid \xi(\{i\}) \gtrless \xi(\phi)\}.$$

A *hedron* \mathfrak{B} in $\mathfrak{S}_0(E)$ is a compact non-empty subset of $\mathfrak{S}_0(E)$ satisfying the following

Exchange axiom for bases [1]. If $\xi, \eta \in \mathfrak{B}$ and $\xi \neq \eta$, then $\xi(\phi) = \eta(\phi)$ and $\exists i \in \text{car}^-(\xi - \eta)$, $\exists j \in \text{car}^+(\xi - \eta)$, $\exists \hat{c} > 0$;

$$(0 \leq) \forall c \leq \hat{c}: \xi + c(\mu^i - \mu^j), \eta + c(\mu^j - \mu^i) \in \mathfrak{B}.$$

We say the modular function $\xi \in \mathfrak{B}$ is a *base* of \mathfrak{B} . Since the Exchange axiom for bases has a self-dual form, it is obvious that if \mathfrak{B} is a hedron in $\mathfrak{S}_0(E)$ then $-\mathfrak{B} \equiv \{-\xi \mid \xi \in \mathfrak{B}\}$ is also a hedron in $\mathfrak{S}_0(E)$, called the *inverse* of \mathfrak{B} .

The *hyperspace* (or *hedron space*) $\mathfrak{H}(E)$ is the linear space of all hedra in $\mathfrak{S}_0(E)$ such that the *sum* is defined by

$$\mathfrak{A} + \mathfrak{B} \equiv \{\xi + \eta \mid \xi \in \mathfrak{A}, \eta \in \mathfrak{B}\}$$

and *scalar multiple* is defined by

$$c\mathfrak{B} \equiv \{c\eta \mid \eta \in \mathfrak{B}\} \quad (c \in \mathbb{R}).$$

Obviously, $\mathfrak{S}_0(E)$ is a subspace of $\mathfrak{F}(E)$. Thus a hyperspace can be regarded as a natural generalization of a vector space.

Let $\mathfrak{S}_\pm(E)$ denote the convex cone of real-valued *super[sub]-modular functions* on 2^E . The *deficiency [rank] function* σ_\pm of a hedron \mathfrak{B} is defined by

$$\sigma_\pm(X) \equiv \min_{\max} \{ \xi(X) \mid \xi \in \mathfrak{B} \}.$$

Then we have the following

Theorem 1. The deficiency [rank] function σ_\pm of a hedron \mathfrak{B} in $\mathfrak{F}(E)$ is a super[sub]modular function, that is $\sigma_\pm \in \mathfrak{S}_\pm(E)$.

The remarkable property of Theorem 1 is that its converse is also true.

Theorem 2. For any super[sub]modular function $\sigma_\pm \in \mathfrak{S}_\pm(E)$, the convex polyhedron \mathfrak{B}_\pm in $\mathfrak{S}_0(E)$ defined by

$$\mathfrak{B}_\pm \equiv \{ \xi \mid \xi \gtrless \sigma_\pm, \xi(E) = \sigma_\pm(E), \xi(\emptyset) = \sigma_\pm(\emptyset) \}$$

is a hedron in $\mathfrak{F}(E)$.

We have then established a one-to-one correspondence between the hedra in $\mathfrak{F}(E)$ and the super[sub]modular functions in $\mathfrak{S}_\pm(E)$.

The concept of *hypermatroids* may be defined in several different but equivalent ways. That is, a hypermatroid \mathfrak{M} is a

pair (E, \mathfrak{B}) , (E, σ_+) , (E, σ_-) , etc., where E is a ground set, \mathfrak{B} is a hedron in $\mathfrak{F}(E)$, and $\sigma_{\pm} (\in \mathfrak{S}_{\pm}(E))$ is the deficiency [rank] function of \mathfrak{B} .

The **-dual* of $\sigma_{\pm} \in \mathfrak{S}_{\pm}(E)$ is defined by

$$\sigma_{\pm}^*(X) \equiv -\sigma_{\pm}(\bar{X}) + \sigma_{\pm}(E) + \sigma_{\pm}(\phi).$$

Theorem 3. The **-dual* of the deficiency [rank] function of a hedron \mathfrak{B} in $\mathfrak{F}(E)$ is the rank [deficiency] function of \mathfrak{B} , that is $\sigma_{\pm}^* = \sigma_{\mp}$.

Hereafter, for notational simplicity let ρ be a submodular function on 2^E . Define a *least upper vector* $\hat{\rho}$ and a *greatest lower vector* $\check{\rho}$ of ρ by

$$\begin{aligned} \hat{\rho}(x) &\equiv \rho(\{x\}) - \rho(\phi) \quad (\forall x \in E), \\ \check{\rho}(x) &\equiv \rho(E) - \rho(E - \{x\}) \quad (\forall x \in E), \end{aligned}$$

respectively. And define a *least upper modular function* $\hat{\rho}$ and a *greatest lower modular function* $\check{\rho}$ of ρ by

$$\begin{aligned} \hat{\rho}(X) &\equiv \sum \hat{\rho}(x) \quad (x \in X), \\ \check{\rho}(X) &\equiv \sum \check{\rho}(x) \quad (x \in X), \end{aligned}$$

respectively. We say $\rho^{\circ} \equiv \hat{\rho} - \check{\rho}$ ($\in \mathfrak{S}_0(E)$) is the *oscillation* of ρ . The *v-dual* of ρ is defined by $\rho^v \equiv \rho^{\circ} - \rho$ and the *h-dual* of ρ is defined by $\rho^h \equiv (\rho^v)^*$. Then we call the hypermatroid $\mathfrak{M}^h =$

(E, ρ^h) defined by a rank function the *dual* of $\mathfrak{M} = (E, \rho)$. The hedron \mathfrak{B}^h of \mathfrak{M}^h is given by

$$\mathfrak{B}^h \equiv \{\rho^o - \xi \mid \xi \in \mathfrak{B}\},$$

and the deficiency function of \mathfrak{M}^h is given by ρ^b . Obviously we have

$$(\mathfrak{M}^h)^h = \mathfrak{M}.$$

Note that the duality of hypermatroids is slightly different from the duality of matroids.

Theorem 4. Any submodular function ρ can be decomposed into three parts as follows:

$$\rho = \rho(\phi)\mu^\phi + \check{\rho} + \tilde{\rho}. \quad (1)$$

We call $\tilde{\rho}$ in (1) the *proper* submodular function of ρ . Let $\tilde{\mathfrak{S}}_-(E)$ denote the set of all proper submodular functions in $\mathfrak{S}_-(E)$.

A hypermatroid $\mathfrak{M} = (E, \rho)$ is called *integral* if ρ is integer-valued. A *polymatroid* [8] $\mathfrak{M} = (E, \rho)$ is a hypermatroid satisfying $\rho(\phi) = 0$ and $\hat{\rho}(x) \geq 0$ ($\forall x \in E$). A *matroid* is an integral polymatroid satisfying $\hat{\rho}(x) \leq 1$ ($\forall x \in E$).

A *quasimatroid* [7] is an integral polymatroid such that its rank function is the direct sum of proper submodular functions

satisfying

$$\left. \begin{aligned} \rho(X \cup \{y\}) + \rho(X \cup \{z\}) - \rho(X) - \rho(X \cup \{y, z\}) &= 0, 1 \\ (y, z \notin X, y \neq z), \end{aligned} \right\} (2)$$

unit functions and the constant function. Obviously a matroid is a quasimatroid.

The following theorem solves the open question by Edmonds [8].

Extreme rays theorem [2]. The extreme rays of $\tilde{\mathcal{E}}_-(E)$ are the proper submodular functions which satisfy the above condition (2) and have minimal sets of intervals $[X, X \cup \{y, z\}]$ such that the left hand side of (2) is equal to 1.

Let $N = (V, A; c)$ be the capacitated network, where V is a vertex set, A is a directed arc set and c is a capacity vector in \mathbb{R}_+^A . Define the *cut function* $\gamma: 2^V \rightarrow \mathbb{R}_+$ by

$$\gamma(X) \equiv c(X, \bar{X}) \equiv \sum c(a) \quad (\delta^+ a \in X, \delta^- a \in \bar{X}, a \in A).$$

For any *flow* $f \in \mathbb{R}_+^A$ satisfying $f \leq c$, define the *boundary function* $\partial f: 2^V \rightarrow \mathbb{R}$ by

$$\partial f(X) \equiv \sum \{f(\delta^+ v) - f(\delta^- v)\} \quad (v \in X).$$

Then γ is submodular and we have $\partial f(\emptyset) = \gamma(\emptyset) = 0$, $\partial f(V) = \gamma(V) = 0$ and $\partial f \leq \gamma$. Therefore, $\mathfrak{R} = (V, \gamma)$ is a hypermatroid defined by

a rank function and every boundary function ∂f for a flow f is a base of \mathfrak{R} [5],[6]. Thus we have known that a capacitated network $N = (V, A; c)$ is a typical example of a hypermatroid.

References

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