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Evolution Equations and Nonlinear Ergodic Theorems

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Let C be a closed convex subset of a real Banach space E. A mapping $T:C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all } x,y \in C.$$

A family $S = \{S(t) : t \geq 0\}$ of mappings on C is said to be a nonexpansive semigroup on C if which satisfies the following conditions:

(i) $S(s+t)x = S(s)S(t)x$ for $s,t \geq 0$ and $x \in C$;

(ii) $\|S(t)x - S(t)y\| \leq \|x - y\|$ for $t \geq 0$ and $x,y \in C$;

(iii) $S(0)x = x$ for $x \in C$;

(iv) $\lim_{t \to t_0} S(t)x = S(t_0)x$ for $t,t_0 \geq 0$ and $x \in C$.

Our purpose is to study the asymptotic behavior of the trajectory $\{T^n x : n \geq 1\}$ of a nonexpansive mapping T and the trajectory $\{S(t)x : t \geq 0\}$ of a nonexpanisve semigroup S. From the study of the asymptotic behavior of the trajectories, we can learn about the asymptotic behavior of the solutions of initial value problem:

$$(1)\quad \begin{cases} \frac{d}{dt}u(t) + Au(t) \ni f(t), & \text{for } t \geqslant 0, \\ u(0) = u_0, & \end{cases}$$

where $A \subset E \times E$ is a m-accretive operator, $f \in L_1(0,\infty:E)$, and $u_0 \in \overline{D(A)}$.

In this paper, we consider the weak convergence , strong convergence and mean convergence of the trajectories. we first study the mean convergence of the trajectories of nonexpansive mappings. we define a term we use.

Definition 1. A sequence $\{x_n\}$ E is said to be weakly almost convergent to a point x in E if

$$\text{weak-}\lim_{n} \frac{1}{n}\left(\sum_{k=0}^{n-1} x_{k+i}\right) = x, \quad \text{uniformly in } i=0,1,2,\cdots$$

Now we introduce the first nonlinear ergodic theorem proved by Baillon[1].

Theorem (Baillon). Let C be a closed convex subset of a real Hilbert space H and $T:C \to C$ be a nonexpansive mapping such that $F(T) = \{z \in C: Tz = z\} \neq \phi$. Then for each $x \in C$, $\{T^n x: n \geqslant 1\}$ is weakly almost convergent to a fixed point of T.

The following corollaries are easy consequences of Baillon's mean ergodic theorem.

Corollary A1. Let C, H and T be as in Theorem A. Then for each $x \in C$, $\{T^n x : n \geq 1\}$ is weakly convergent to a fixed point of T if and only if $\text{weak-}\lim_n (T^{n+1}x - T^n x) = 0$.

Corollary A2. Let C and H be as in Theorem A. Let $S = \{S(t);\ t \geq 0\}$ be a nonexpansive semigroup on C such that $F(S) = \{z \in C : S(t)z = z \text{ for all } t \geq 0\} \neq \phi$. Then for each $x \in C$, there exists $y \in F(S)$ and

$$\text{weak-}\lim_T \frac{1}{T} \int_t^{t+T} S(s)x\, ds = y, \quad \text{uniformly in } t \geq 0.$$

Our purpose is to establish the mean ergodic theorem in Banach spaces. In Banach spaces, it is far difficult to prove the mean ergodic theorem. But recently, S. Reich proved the following result[8].

Theorem B(Reich). Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a closed convex subset of E and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \phi$. Then for each $x \in C$, $\{T^n x : n \geq 1\}$ is weakly almost convergent to a fixed point of T.

In [5] we gave a proof of Theorem B which is slightly different form Reich's. Our proof is based on the following three lemmas.

Lemma 1. Let C be a closed convex subset of a uniformly convex Banach space E and $T:C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$. Then for each $x \in C$ and each positive integer n,

$$\lim_{i} \| T^k S_n T^i x - S_n T^{k+i} x \| = 0,$$

uniformly in $k \geq 1$, where $S_n z = \frac{1}{n} \sum_{k=0}^{n-1} T^k z$ for each $z \in C$.

Lemma 2. Let E,C and T be as in TheoremB. Then for each $x \in C$,

$$\bigcap_{k} \overline{co}\{ T^n x : n \geq k \} \cap F(T)$$

contains exactly one point.

Lemma A(Browder). Let C be a closed convex subset of a uniformly convex Banach space E and $T:C \to E$ be a nonexpansive mapping with $F(T) \neq \phi$. Let $\{x_n\} \subset E$ be a bounded sequence such that $\lim_n \| x_n - Tx_n \| = 0$. Then any subsequential weak limit point of $\{x_n\}$ is a fixed point of T.

We give the sketch of the proof of Theorem B.

Sketch of proof of Theorem B. Let $x \in C$. By Lemma 1, we can construct a sequence $\{S_n T^{k_n} x : n \geq 1\}$ E such that

$$k_{n+1} \geq k_n \quad \text{for } n \geq 1 \quad \text{and} \quad \lim_n \| T S_n T^{k_n} x - S_n T^{k_n} x \| = 0.$$

For simplicity we set $x_n = S_n T^{k_n} x$ for each $n \geq 1$. Then $\lim_n \| x_n - Tx_n \| = 0$. While from Lemma 2, we have

$$\bigcap_K \overline{co} \{ x_n \ ; \ n \geq k \} \cap F(T) = \{y\}.$$

Therefore , by Lemma A, we have that $\{x_n\}$ converges weakly to a fixed point y of T. By a similar argument, we can see that for any sequence $\{h_n\}$ of integer such that $h_n \geq k_n$ for $n \geq 1$, $\{S_n T^{h_n} x\}$ converges weakly to y. While for n and m with $m \geq k_n$,

$$S_m x = \frac{1}{m} \sum_{k=0}^{m-1} T^k x = \frac{1}{m} \Big(\sum_{k=k_n+jn}^{m-1} T^k x + n \Big(\sum_{k=0}^{j-1} S_n T^{k_n+kn} x \Big) + \sum_{k=0}^{k_n} T^k x \Big)$$

where $m = jn + k_n + r$, $r < n$. Since $\{S_n T^{k_n+kn} x\}$ converges weakly to y, we obtain that $S_m x$ converges weakly to y.

By using Lemma 1, we also prove another nonlinear ergodic theorem.

Theorem 1. Let C be a closed convex subset of a uniformly convex Banach space E and $T:C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$. Let P be the metric projection on $F(T)$ and suppose that P has the following property (C):

(C) If $\{x_n\}$ converges weakly to a point x in $F(T)$ and $\{Px_n\}$ converges strongly to y, then $x = Px = y$.

Then for each $x \in C$, $\{T^n x\}$ is almost convergent to a fixed point of T.

The property (C) is determined by the structure of Banach spaces and the shape of the set $F(T)$. For example, we have the following corollary of Theorem 1.

Corollary 1. Let E be a uniformly convex Banach space which satisfies Opial's condition. Let C be a closed convex subset of E and $T:C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$. Then for each $x \in C$, $\{T^n x\}$ is weakly almost convergent to a fixed point of T.

Now we prove Theorem 1 by using Lemma 1 and Lemma A.

Proof of Theorem 1. From Lemma 1, we can construct a sequence $\{S_{2^n}T^{k_n}x\} \subset E$ such that

$$k_{n+1} \geq k_n \quad \text{for } n \geq 1, \quad \lim_n \| TS_{2^n}T^{k_n}x - S_{2^n}T^{k_n}x \| = 0. \qquad (*)$$

Here we set

$$\Sigma = \{\{S_{2^n}T^{k_n}x\} \subset E: \{S_{2^n}T^{k}x\} \text{ satisfies } (*)\}$$

Then for each $\{S_{2^n}T^{h_n}x\} \in \Sigma$ and $y \in F(T)$, $\lim_n \| S_{2^n}T^{h_n}x - y \|$ exists[6]. Also we can see that for $\{S_{2^n}T^{k_n}x\}$, $\{S_{2^n}T^{h_n}x\} \in \Sigma$, with $h_n \geq k_n$ for all $n \geq 1$,

$$\begin{aligned}
&\lim_n \| S_{2^n}T^{h_n}x - y \| \\
&\quad \leq \lim_n \| S_{2^n}T^{h_n}x - T^{h_n-k_n}S_{2^n}T^{k_n}x \| + \lim_n \| T^{h_n-k_n}S_{2^n}T^{k_n}x - y \| \\
&\quad \leq \lim_n \| S_{2^n}T^{k_n+(h_n-k_n)}x - T^{h_n-k_n}S_{2^n}T^{k_n}x \| \\
&\qquad\qquad + \lim_n \| T^{h_n-k_n}S_{2^n}T^{h_n}x - y \| \\
&\quad = \lim_n \| S_{2^n}T^{k_n}x - y \| \qquad \text{for each } y \in F(T). \qquad (**)
\end{aligned}$$

Now we set for each $y \in F(T)$,

$$r(y) = \inf \{\lim_n \| S_{2^n}T^{k_n}x - y \| : \{S_{2^n}T^{k_n}x\} \in \Sigma\},$$

$$r = \inf \{r(y) : y \in F(T) \}.$$

Then we can see that there exists $z \in F(T)$ such that $r(z) = r$ and there exists $\{S_{2^n}T^{k_n}x\} \in \Sigma$ such that $\lim_n \| S_{2^n}T^{k_n}x - z \| = r$.

Next we claim that $\lim_n \| S_{2^n}T^{k_n}x - PS_{2^n}T^{k_n}x \|$ exists and $\{PS_{2^n}T^{k_n}x\}$ converges strongly to z. For given $\varepsilon > 0$, let n_0 be a positive integer such that

$$\| S_{2^{n_0}}T^{k_{n_0}}x - PS_{2^{n_0}}T^{k_{n_0}}x \| \leq \liminf_n \| S_{2^n}T^{k_n}x - PS_{2^n}T^{k_n}x \| + \varepsilon/2.$$

and

$$\| S_{2^{n_0}}T^{k_{n_0}}x - PS_{2^{n_0}}T^{k_{n_0}}x \| \leq \lim_n \| S_{2^n}T^{k_n}x - PS_{2^{n_0}}T^{k_n}x \| + \varepsilon/2.$$

Then we have that

$$\limsup_n \| S_{2^n}T^{k_n}x - PS_{2^n}T^{k_n}x \| \leq \lim_n \| S_{2^n}T^{k_n}x - PS_{2^{n_0}}T^{k_{n_0}}x \|$$

$$\leq \| S_{2^{n_0}}T^{k_n}x - PS_{2^n}T^{k_n}x \| + \varepsilon.$$

Therefore we obtain that $\lim_n \| S_{2^n}T^{k_n}x - PS_{2^n}T^{k_n}x \|$ exists and from this fact, we can see that $\{PS_{2^n}T^{k_n}x\}$ converges strongly to z. Hence from the property (C), we have that $\{S_{2^n}T^{k_n}x\}$ converges weakly to $z \in F(T)$. While from the inequality (**), we obtain that

$$\lim_n \| S_{2^n}T^{h_n}x - z \| = z,$$

for all $\{S_{2^n}T^{h_n}x\} \in \Sigma$ with $h_n \geq k_n$ for $n \geq 1$. Moreover, we can see that

$$\text{weak-}\lim_n S_{2^n}T^{h_n}x = z,$$

uniformly in $\{S_{2^n} T^{h_n} x\} \in \Sigma$ with $h_n \geqslant k_n$ for $n \geqslant 1$. Then by the same argument as in the proof of Theorem B, we obtain that $\{T^n x\}$ is weakly almost convergent to a fixed point of T.

From Theorem 1, we can deduce the following corollary.

Corollary 2. Let C be a closed convex subset of a uniformly convex Banach space E and $T:C \longrightarrow C$ be a mapping which satisfies the following condition:

$$\| Tx - Ty \| \leqslant \| r(x - y) + (1-r)(Tx - Ty) \|$$

for all $x,y \in C$ and all $r > 0$. Suppose that $F(T) \neq \phi$ and the metric projection P on F(T) satisfies the property (C). Then for each $x \in C$, $\{T^n x\}$ converges weakly to a fixed point of T.

Remark. A mapping T which satisfies the condition in Corollary 2 is said to be firmly nonexpansive. It is easy to see that a mapping T is nonexpansive if T is firmly nonexpansive.

Next we refer to the strong convergent of the trajectories of nonexpansive mappings.

Theorem 2. Let C be a closed convex subset of a uniformly convex Banach space E and $T:C \to C$ be a nonexpansive mapping. Suppose that the set F(T) has a interior point. Then for each $x \in C$, $\{T^n x\}$ converges strongly to a fixed point of T.

Proof. Let $x \in C$ and z be a interior point of F(T). Then there exists $r > 0$ such that

$$S_r(z) = \{y \in C : \| y - z \| \leq r \} \subset F(T).$$

Let P be the metric projection on $S_r(z)$. Then we claim that $\{PT^n x\}$ converges strongly to a point x_0 in F(T). Since T is nonexpansive, the sequence $\{\|T^n x - PT^n x\|\}$ is monotone decreasing. Suppose that there exists a subsequence $\{PT^{n_i} x\}$ of $\{PT^n x\}$ such that $\| PT^{n_i} x - PT^{n_{i+1}} x \| \geq \varepsilon$ for some $\varepsilon > 0$. Let $r = \lim_n \| T^n x - PT^n x \|$. Choose $c > 0$ so small that $r > (r+c)(1-\delta(\varepsilon/(r+c))$, where δ is the modulus of convexity of the norm of E. Then for each $i \geq 1$ such that $\| T^{n_i} x - PT^{n_i} x \| \leq r+c$, we have that

$$\begin{aligned} \| T^{n_{i+1}} x - PT^{n_{i+1}} x \| &\leq \| T^{n_{i+1}} x - (PT^{n_i} x + PT^{n_{i+1}} x)/2 \| \\ &\leq (r+c)(1 - \delta(\varepsilon/(r+c)) \\ &< r. \end{aligned}$$

This is a contradiction. Therefore $\{PT^n x\}$ converges strongly to a fixed point of T.

Then we can conclude that $\{T^n x\}$ converges strongly to a fixed point of T since $PT^n x \in \{az + (1-c)T^n x : 0 \leq a \leq 1\}$.

From this theorem, we can deduce the following theorem.

Theorem 3. Let C be a closed convex subset of a uniformly convex Banach space E and S be a nonexpansive semigroup on C such that F(S) has a interior point. Then for each $x \in C$, $\{S(t) : t \geq 0\}$ converges strongly to a common fixed point of S.

Remark 2. Theorem 3 is due to Brezis in the case E is a Hilbert space.

Remark 3. The uniformly convexity of the norm of Banach spaces is essential for our method employed in the proof of Theorem 1, Theorem 2 and Theorem 3. But we do not know whether the Fréchet differentiability of norms or the property (C) of the metric projection is essential or not.

Remark 4. It is pointed out by Brezis that proved Theorem 2 under the same condition.

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