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On the strong convergence of the Cesàro means of contractions in Banach spaces, II

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1. Introduction. Throughout this note X denotes a uniformly convex Banach space and C is a nonempty closed convex subset of X. A mapping $T: C \to C$ is called a contraction on C, or $T \in \mathrm{Cont}(C)$ if $\| Tx - Ty \| \le \| x - y \|$ for every $x, y \in C$. A family $\{T(t): t \ge 0\}$ of mappings from C into itself is called a contraction semigroup on C if $T(0) = I$, $T(t + s) = T(t)T(s)$, $T(t) \in \mathrm{Cont}(C)$ for $t, s \ge 0$ and $\lim_{t \downarrow 0} \| T(t)x - x\| = 0$ for every $x \in C$. The set of fixed points of a mapping T will be denoted by F(T). We set

$$S_n x = n^{-1} \sum_{i=0}^{n-1} T^i x \quad \text{and} \quad \sigma_t x = t^{-1} \int_0^t T(s)x\, ds$$

for $x \in C$, $n = 1,2,\cdots$ and $t > 0$.

The purpose of this note is to prove the following theorems which are obtained by the author and Miyadera (see [8]).

Theorem 1. Let $T \in \mathrm{Cont}(C)$ and $x \in C$. The following (a) and (b) are equivalent:

(a) There exists an element y of F(T) such that (the strong) $\lim_{n \to \infty} S_n T^k x = y$ uniformly in $k = 0,1,2,\cdots$.

(b) $F(T) \ne \emptyset$ and

$$(1) \quad \lim_{n,m \to \infty} \| 2^{-1}(S_n T^{\ell+n} x + S_m T^{\ell+m} x) - T^{\ell}(2^{-1} S_n T^n x + 2^{-1} S_m T^m x) \| = 0$$

holds uniformly in $\ell = 1,2,\cdots$.

Theorem 2. Let $\{T(t): t \geq 0\}$ be a contraction semigroup on C and $x \in C$. The following (a)' and (b)' are equivalent:

(a)' There exists an element y of $\bigcap_{t>0} F(T(t))$ such that (the strong) $\lim_{t \to \infty} \sigma_t T(h)x = y$ uniformly in $h > 0$.

(b)' $\bigcap_{t>0} F(T(t)) \neq \emptyset$ and

$$(1)' \qquad \lim_{s,t \to \infty} \| 2^{-1}(\sigma_t T(t+h)x + \sigma_s T(s+h)x) - T(h)(2^{-1}\sigma_t T(t)x + 2^{-1}\sigma_s T(s)x) \| = 0$$

holds uniformly in $h > 0$.

Remark 1. Let $T \in \mathrm{Cont}(C)$ and $\{T(t): t \geq 0\}$ be a contraction semigroup on C. If $F(T) \neq \emptyset$ (resp. $\bigcap_{t>0} F(T(t)) \neq \emptyset$), we have the limit (1) (resp. (1)') for each $\ell = 1,2,\cdots$ (resp. $h > 0$).

2. Proofs of Theorems. We first note that for every sequence $\{x_n\}$ in X the following equality holds: For any ℓ, $p \geq 1$ and $k \geq 0$

$$(2) \qquad \ell^{-1} \sum_{i=0}^{\ell-1} x_{i+k} = \ell^{-1} \sum_{i=0}^{\ell-1} (p^{-1} \sum_{j=0}^{p-1} x_{j+i+k}) + (\ell p)^{-1} \sum_{i=1}^{p-1} (p-i)(x_{i+k-1} - x_{i+k+\ell-1})$$

Lemma 1. Let $T \in \mathrm{Cont}(C)$ and $x \in C$. If (b) of Theorem 1 is satisfied, then $\{\| S_n T^n x - f \|\}$ is convergent for every $f \in F(T)$.

Proof. Let $f \in F(T)$ and $\alpha_n = \sup_{j \geq 0} \| S_n T^{n+j} x - T^j S_n T^n x \|$ for $n \geq 1$. Since

$$S_{n+m} T^{n+m} x = (n+m)^{-1} \sum_{i=0}^{n+m-1} (S_n T^{n+m+i} x - T^{m+i} S_n T^n x) + (n+m)^{-1} \sum_{i=0}^{n+m-1} T^{m+i} S_n T^n x$$

$$+ [n(n+m)]^{-1} \sum_{i=1}^{n-1} (n - i)[T^{n+m+i-1}x - T^{2(n+m)+i-1}x]$$

by (2), we have

$$\| S_{n+m}T^{n+m}x - f\| \leq \alpha_n + (n+m)^{-1} \sum_{i=0}^{n+m-1} \| T^{m+i}S_nT^nx - f \|$$

$$+ [n(n+m)]^{-1} \sum_{i=1}^{n-1} (n - i) \| T^{n+m+i-1}x - T^{2(n+m)+i-1}x\|$$

$$\leq \alpha_n + \| S_nT^nx - f\| + (n-1)\| x - f\| /(n+m)$$

for n, m $\geq$ 1. Letting $m \to \infty$, we get

$$\limsup_{m \to \infty} \| S_mT^mx - f \| \leq \alpha_n + \| S_nT^nx - f\|$$

for $n \geq 1$. Since taking n = m in (1) yields that $\lim_{n\to\infty} \alpha_n = 0$, we obtain that $\limsup_{m \to \infty} \| S_mT^mx - f\| \leq \liminf_{n \to \infty}\| S_nT^nx - f\|$.

Q.E.D.

Lemma 2. Let $T \in \mathrm{Cont}(C)$ and $x \in C$. If (b) of Theorem 1 is satisfied, then there exists an element y of F(T) such that $\lim_{n \to \infty} S_nT^{n+k}x = y$ uniformly in k = 0,1,2,···.

Proof. Take an $f \in F(T)$ and set $u_n = S_nT^nx - f$ for $n \geq 1$. By Lemma 1, $d = \lim_{n \to \infty} \| u_n \|$ exists. Since $\| u_{n+1} - u_n \| \to 0$ as $n \to \infty$, we have

(3) $$\lim_{n \to \infty} \| u_n + u_{n+i} \| = 2d \qquad \text{for every } i \geq 0.$$

We now show that $\{S_nT^nx\}$ is strongly convergent to an element of F(T). To this end set

$$v(n,k) = [n(n+m)]^{-1} \sum_{i=1}^{n-1} [T^{n+k+i-1}x - T^{2(n+m)+i-1}x].$$

since $S_{n+k}T^{n+k}x = (n+k)^{-1} \sum_{i=0}^{n+k-1} S_n T^{n+k+i}x + v(n,k)$ by (2) and $\| v(n,k)\| \leq (n-1)\| x - f \|/(n+k)$, we have

$$\| u_{n+k} + u_{m+k} \|$$

$$= \| (n+k)^{-1} \sum_{i=0}^{n+k-1} (S_n T^{n+k+i}x + S_m T^{m+k+i}x - 2f)$$

$$+[(n-m)/(m+k)(n+k)] \sum_{i=0}^{n+k-1} (S_m T^{m+k+i}x - f)$$

$$+ (m+k)^{-1} \sum_{i=n+k}^{m+k-1} (S_m T^{m+k+i}x - f) + v(n,k) + v(m,k)\|$$

$$\leq [2/(n+k)] \sum_{i=0}^{n+k-1} \| 2^{-1}(S_n T^{n+k+i}x + S_m T^{m+k+i}x) - f \|$$

$$+ \| x - f \| [(m-n)/(m+k) + (n-1)/(m+k) + (m-1)/(m+k)]$$

for $m \geq n \geq 1$ and $k \geq 0$. Combining this with $\| 2^{-1}(S_n T^{n+k+i}x + S_m T^{m+k+i}x) - f \| \leq \alpha_{n,m} + \| 2^{-1}(S_n T^n x + S_m T^m x) - f \|$, where $\alpha_{n,m} = \sup_{\ell \geq 0} \| 2^{-1}(S_n T^{\ell+n}x + S_m T^{\ell+m}x) - T^{\ell}(2^{-1}S_n T^n x + 2^{-1}S_m T^m x)\|$, we obtain

$$\| u_{n+k} + u_{m+k} \|$$

$$\leq 2\alpha_{n,m} + \| u_n + u_m \| + \| x - f \| \left[\frac{2(m-n)}{m+k} + \frac{n-1}{n+k} + \frac{m-1}{m+k} \right]$$

for $m \geq n \geq 1$ and $k \geq 0$. Letting $k \to \infty$, we get from (3) that

$$2d \leq 2\alpha_{n,m} + \| u_n + u_m \|$$

$$\leq 2\alpha_{n,m} + \| u_n \| + \| u_m \|$$

for all $n,m \geq 1$. Since $\lim_{n,m\to\infty} \alpha_{n,m} = 0$ by (i), we have that $\lim_{n,m\to\infty} \| u_n + u_m \| = 2d$. By uniform convexity of X and $\lim_{n\to\infty} \| u_n \|$

$= d$, $\lim_{n,m\to\infty} \| S_nT^nx - S_mT^mx \| = \lim_{n,m\to\infty} \| u_n - u_m \| = 0$, whence $\{S_nT^nx\}$ converges strongly. Put $y = \lim_{n\to\infty} S_nT^nx$. By (1) with $n = m$, $\| S_nT^nx - TS_nT^nx \| \le 2\| x - f \| /n + \| S_nT^{n+1}x - TS_nT^nx \| \to 0$ as $n \to \infty$, and hence $y \in F(T)$.

Finally, by (1) with $n = m$ again,

$$\sup_{k\ge 0} \| S_nT^{n+k}x - y \|$$
$$\le \sup_{k\ge 0} \| S_nT^{n+k}x - T^kS_nT^nx \| + \| S_nT^nx - y \| \to 0 \quad \text{as } n \to \infty.$$

Q.E.D.

Proof of Theorem 1. We first assume that (b) holds. By virtue of Lemma 2, there exists an element y of F(T) such that $\lim_{n\to\infty} S_nT^{n+k}x = y$ uniformly in $k \ge 0$. Therefore, for any $\varepsilon > 0$ there exists a positive integer N such that $\| S_NT^{N+j}x - y \| \le \varepsilon$ for any $j \ge 0$. Since

$$S_nT^kx = n^{-1} \sum_{i=0}^{n-1} S_NT^{k+i}x + (nN)^{-1} \sum_{i=1}^{N-1} (N - i)(T^{k+i-1}x - T^{k+i+n-1}x)$$

by (2), for $n > N$ we have

$$\| S_nT^kx - y \| \le n^{-1} \sum_{i=0}^{n-1} \| S_NT^{k+i}x - y \| + (N - 1)\| x - y \| /n$$
$$\le n^{-1} \sum_{i=0}^{N-1} \| S_NT^{k+i}x - y \| + n^{-1} \sum_{i=N}^{n-1} \| S_NT^{k+i}x - y \|$$
$$+ (N - 1)\| x - y \| /n$$
$$\le N\| x - y \| /n + \varepsilon + (N - 1)\| x - y \| /n.$$

Hence $\sup_{k\ge 0} \| S_nT^kx - y \| \to 0$ as $n \to \infty$. This proves that (a) holds.

Next, assume that (a) holds. Put $y = \lim_{n\to\infty} S_nx$. Since $y \in F(T)$, F(T) in nonempty. Moreover,

$$\| 2^{-1}(S_n T^{n+\ell}x + S_m T^{m+\ell}x) - T^{\ell}(2^{-1}S_n T^n x + 2^{-1}S_m T^m x)\|$$

$$\leq \| 2^{-1}(S_n T^{n+\ell}x + S_m T^{m+\ell}x) - y \| + \| (2^{-1}S_n T^n x + 2^{-1}S_m T^m x) - y\|$$

$$\leq 2^{-1}\{\| S_n T^{n+\ell}x - y \| + \| S_m T^{m+\ell}x - y\| + \| S_n T^n - y\| + \|S_m T^m x - y\|\}.$$

Since the right hand of the above inequality goes to 0 as $n \to \infty$ uniformly in $\ell \geq 1$, (1) holds uniformly in $\ell \geq 1$. Q.E.D.

Proof of Theorem 2. First, assume that (b)' holds. Similarly as in the proof of the preceding lemmas, we have the following (c) and (d):

(c) $\lim_{t\to\infty} \| \sigma_t T(t) - f \|$ exists for every $f \in \bigcap_{t>0} F(T(t))$.

(d) There exists an element y of $\bigcap_{t>0} F(T(t))$ such that $\lim_{t\to\infty} \sigma_t T(t+h)x = y$ uniformly in $h \geq 0$.

To prove (c) and (d) we use the following equality instead of (2):

$$t^{-1} \int_0^t T(\xi+h)x \, d\xi = t^{-1} \int_0^t [s^{-1} \int_0^s T(\xi+\eta+h)x \, d\eta] \, d\xi$$
$$+ (ts)^{-1} \int_0^s (s - \eta)[T(\eta+h)x - T(\eta+t+h)x] \, d\eta$$

for $t, s > 0$ and $h \geq 0$. Now, the same argument as in the proof of Theorem 1 implies that $\lim_{t\to\infty} \sigma_t T(h)x = y$ uniformly in $h \geq 0$, which shows that (b)' implies (a)'. Conversely, we can obtain that (a)' implies (b)' as in the same argument as in the proof of Theorem 1. Q.E.D.

3. Corollaries. In this section we shall consider sufficient conditions for (b) and (b)'.

Lemma 3. Let $T \in \mathrm{Cont}(C)$, $x \in C$ and $F(T) \neq \emptyset$. Suppose:

(i) T is affine on C, or

(ii) $\lim_{n\to\infty} \| T^n x - T^{n+i} x \|$ ($= \beta(i)$) exists uniformly in $i \geq 1$.

Then we have condition (b) of Theorem 1.

Proof. If T is affine on C, then it is evident that (b) holds, and hence assume condition (ii). Take an $f \in F(T)$ and an $r > 0$ with $r \geq \| x - f \|$, and set $D = \{z \in X: \| z - f \| \leq r\} \cap C$ and $U = T|_D$. Since D is nonempty bounded closed convex and $U \in \mathrm{Cont}(D)$, by virtue of [7, Theorem 2.1] (cf. [6, Lemma 1.1]) there exists a strictly increasing continuous convex function $\gamma: [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that

$$\| U^\ell (\sum_{i=1}^{k} \lambda_i x_i) - \sum_{i=1}^{k} \lambda_i U^\ell x_i \|$$

$$\leq \gamma^{-1}(\max_{1 \leq i,j \leq k} [\| x_i - x_j \| - \| U x_i - U x_j \|]$$

for any $\lambda_1, \cdots, \lambda_k \geq 0$ with $\lambda_1 + \cdots + \lambda_k = 1$, any $x_1, \cdots, x_k \in D$ and any k, $\ell \geq 1$. Consequently,

$$\| T^\ell (\sum_{i=0}^{n-1} \lambda_i x_i + \sum_{i=0}^{m-1} \mu_i y_i) - (\sum_{i=0}^{n-1} \lambda_i T^\ell x_i + \sum_{i=0}^{m-1} \mu_i T^\ell y_i) \|$$

$$\leq \gamma^{-1}(\max\{ \| x_i - y_j \| - \| T^\ell x_i - T^\ell x_j \| , \| x_i - x_p \| - \| T^\ell x_i - T^\ell x_p \| ,$$

$$\| y_p - y_q \| - \| T^\ell y_p - T^\ell y_q \| ; \ 0 < i,j < n-1, \ 0 < p,q < m-1\})$$

for any λ_i, $\mu_i \geq 0$ with $\sum_{i=0}^{n-1} \lambda_i + \sum_{i=0}^{m-1} \mu_i = 1$, any x_i, $y_i \in D$ and any n, $m \geq 1$, $\ell \geq 0$. Using this inequality with $\lambda_i = 1/2n$, $x_i = T^{i+n} x$ for $0 \leq i \leq n-1$ and $\mu_i = 1/2m$, $y_i = T^{i+m} x$ for $0 \leq i \leq m-1$, we obtain

$$(4)\quad \| T^{\ell}(2^{-1}S_nT^nx + 2^{-1}S_mT^mx) - (2^{-1}S_nT^{\ell+n}x + 2^{-1}S_mT^{\ell+m}x) \|$$

$$\leq \gamma^{-1}(\max\{\| T^{i+n}x - T^{j+n}x \| - \| T^{\ell+i+n}x - T^{\ell+j+n}x \| ,$$

$$\| T^{i+n}x - T^{p+n}x \| - \| T^{\ell+i+n}x - T^{\ell+p+n}x \| ,$$

$$\| T^{p+m}x - T^{q+m}x \| - \| T^{\ell+p+m}x - T^{\ell+q+m}x \| :$$

$$0 \leq i,j \leq n-1,\ 0 \leq p,q \leq m-1\})$$

for any $n, m \geq 1$ and $\ell \geq 0$.

For any $\varepsilon > 0$ choose a $\delta > 0$ such that $\gamma^{-1}(\delta) < \varepsilon$. By (ii) there exists a positive integer N such that $\beta(i) \leq \| T^nx - T^{n+i}x \| < \beta(i) + \delta$ for every $i \geq 0$ and $n \geq N$. Hence if $n, m \geq N$, then

$$\| T^{i+n}x - T^{j+m}x \| - \| T^{\ell+i+n}x - T^{\ell+j+m}x \|$$

$$< \beta(|j+m-i-n|) + \delta - \beta(|j+m-i-n|) = \delta$$

for every $i, j \geq 0$. Combining this with (4), we obtain that if $n, m \geq N$, then

$$\| T^{\ell}(2^{-1}S_nT^nx + 2^{-1}S_mT^mx) - (2^{-1}S_nT^{\ell+n}x + 2^{-1}S_mT^{\ell+m}x) \|$$

$$\leq \gamma^{-1}(\delta) < \varepsilon \qquad \text{for any } \ell \geq 0.$$

Thus (1) holds uniformly in $\ell > 0$. Q.E.D.

Similarly as in the proof of the above lemma, we have the following

Lemma 4. Let $\{T(t): t \geq 0\}$ be a contraction semigroup on C, $x \in C$ and $\bigcap_{t>0} F(T(t)) \neq \emptyset$. Suppose (i)' each $T(t)$ is affine on C, or

(ii)' $\lim_{t\to\infty} \|T(t)x - T(t+h)x\|$ $(=\beta'(h))$ exists uniformly in $h > 0$.

Then we have condition (b)' of Theorem 2.

Combining Theorems 1 and 2 with Lemmas 3 and 4 respectively, we have the following corollaries.

Corollary 5. Let $T \in \mathrm{Cont}(C)$, $x \in C$ and $F(T) \neq \emptyset$. If one of conditions (i) and (ii) of Lemma 3 is satisfied, then there exists an element y of $F(T)$ such that $\lim_{n\to\infty} S_n T^k x = y$ uniformly in $k \geq 0$.

Corollary 6. Let $\{T(t): t \geq 0\}$ be a contraction semigroup on C, $x \in C$ and $\bigcap_{t>0} F(T(t)) \neq \emptyset$. If one of conditions (i)' and (ii)' of Lemma 4 is satisfied, then there exists an element y of $\bigcap_{t>0} F(T(t))$ such that $\lim_{t\to\infty} \sigma_t T(h)x = y$ uniformly in $h \geq 0$.

Remarks 2. 1) Let $T \in \mathrm{Cont}(C)$ and $x \in C$. If $\{T^n x\}$ has a convergent subsequence, then condition (ii) of Lemma 3 is satisfied (cf. [5, Theorem 2.4]). 2) Let X be a Hilbert space, and let $T \in \mathrm{Cont}(C)$. If T is odd, then condition (ii) of Lemma 3 is satisfied for every $x \in C$ (cf. [4]).

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