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CAPILLARY SURFACES IN THE ABSENCE OF GRAVITY; THE TRAPEZOIDAL CAPILLARY TUBE

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1. Consider a cylindrical container Z of homogeneous composition, partly filled with a volume V of liquid, and situated in outer space in the absence of gravity (fig 1). According to the Principle of Virtual Work, the free surface S will be determined by the condition that the potential energy of the system is stationary with respect to all variations consistent with the constraints. We assume a surface S in the form $z = u(x,y)$ over the base Ω ; the energy - up to multiplicative and additive constants and modified by a volume constraint term with Lagrange parameter 2H - then takes the form

(1) $$E = S - \beta Q + 2HV$$

where β is a constant corresponding to the attractive force, over the wetted surface Q, between the liquid and the container. Here and in what follows, symbols such as S, Q, V will be used alternatively, to denote both a set and its measure.

The variational principle leads to the relations

(2) $$\text{div}\, Tu = 2H \quad \text{in } \Omega$$

with

(3) $$Tu = \frac{1}{\sqrt{1 + |\nabla u|^2}} \nabla u \quad ,$$

and

(4) $$\nu \cdot Tu = \beta \quad \text{on } \Sigma = \partial\Omega \quad ,$$

where ν is the unit exterior normal on Σ. Denoting by γ the angle between S and Z on the manifold of contact, we find $\nu \cdot Tu = \cos \gamma$. Thus, we may rewrite (4) in the form

(5) $$\nu \cdot Tu = \cos \gamma \equiv \text{const} \quad \text{on } \Sigma.$$

We note from (2), (3) that H is exactly the mean curvature of the surface S, taken as positive when the surface curves upward. Integrating (2) over Ω yields

(6) $$2H = \frac{\Sigma}{\Omega} \cos \gamma$$

and thus (2) takes the form

(7) $$\operatorname{div} Tu = \frac{\Sigma}{\Omega} \cos \gamma \quad \text{in } \Omega$$

The constant γ is determined by the physical properties of the materials. Once it is known, the problem becomes a geometrical one : to determine a surface of prescibed constant mean curvature that meets prescribed boundary walls in a prescribed constant angle. Alternatively the problem can be viewed as that of finding a solution of a certain nonlinear elliptic equation under nonlinear boundary conditions.

In what follows we assume $0 \leq \gamma \leq \frac{\pi}{2}$; the remaining case reduces to that one under the transformation $u = -v$.

It was pointed out in [1] that the problem (5), (7) need not always have a solution. Consider an arbitrary curve Γ, which together with a subset $\Sigma^* \subset \Sigma$ bounds a subdomain $\Omega^* \subset \Omega$ (fig 2). Integration of (7) over Ω^* yields

$$(8)\qquad \left(\frac{\Sigma}{\Omega}\,\Omega^* - \Sigma^*\right)\cos\gamma = \int_\Gamma \nu \cdot Tu\,ds$$

But for *any* differentiable function f(x,y) there holds

$$(9)\qquad |Tf| = \frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}} < 1$$

We have proved :

A necessary condition for the existence of a solution of (5), (7) *in* Ω *is that*

$$(10)\qquad \varphi(\Gamma) \equiv \left(\frac{\Sigma}{\Omega}\,\Omega^* - \Sigma^*\right)\cos\gamma + \Gamma > 0 \,.$$

for every Γ *of the type considered.*

Let us examine this result in the special case in which Ω contains a corner of opening 2α (fig 3). We find, for the indicated Γ,

$$(11)\qquad \varphi(\Gamma) = 2l(\sin\alpha - \cos\gamma) + (l^2).$$

Letting $l \to 0$, we obtain immediately the result that *if* $\alpha + \gamma < \pi/2$ *there is no solution to the indicated problem.*

The difficulty does not arise from the boundary discontinuity at V. One sees easily that the same contradiction can be obtained when the boundary is smoothed at V. Thus we are faced with an elliptic boundary value problem that arises directly from physical considerations, which is in general not well-posed, even in a smooth convex domain.

The condition we have found is remarkable in that it is sharp. Let Σ be a regular polygon and C the circumscribed circle(fig 4). A lower hemisphere through C provides an explicit solution of (5), (7) for the case $\alpha + \gamma = \pi/2$. The solution is analytic up to the polygonal walls and continuous in the closed region. Replacing the hemispere by spherical caps of increasing radius, we find explicit solutions of the problem for any γ satisfying $\alpha + \gamma \geq \pi/2$. Thus, in general *the solutions to* (5), (7) *in a given domain depend discontinuously on the boundary datum* γ. In the example cited, as γ decreases from $\pi/2$ to $\frac{\pi}{2} - \alpha$, the solutions exist and remain uniformly bounded and analytic in Ω. But if $\gamma < \frac{\pi}{2} - \alpha$, no solution exists.

The behaviour just described was verified experimentally by NASA in a "drop tower", which provides about five seconds of "free fall" without gravity. If $\alpha + \gamma \geq \pi/2$ the spherical cap solution is obtained; if $\alpha + \gamma < \pi/2$ the fluid flows up into the corners to infinity or to the top of the container, whichever comes first.

We point out here that *whenever a solution exists, it is uniquely determined up to an additive constant.*

2. We wish to characterize geometrically the conditions under which a solution will exist. Some such (necessary) conditions are given in [1, Theorem 3]. Giusti [2] obtained the basic theorem that *if* (10) *holds for every subdomain* Ω^*, *then a solution exists.* Chen [3] considered the case $\gamma = 0$. He showed that *if a disk of radius* $R_0 = \frac{\Omega}{\Sigma}$ *can be rolled around* Σ *interior to* Ω, *then a solution exists.*
The condition is sufficient but not necessary, as Chen showed by example. In seeking necessary conditions, Chen introduced the notions of "neck domain" and "tail domain". A subdomain $\Omega^* \subset \Omega$ as above is called a "tail domain" if Γ is a circular arc of radius $\frac{\Omega}{\Sigma}$ that meets Σ tangentially and if there is no other such arc Γ interior to Ω^* (fig 5). Chen showed that *if* Ω *contains a tail domain, then no solution exists for* $\gamma = 0$.

In [4] the case of general γ is considered and it is shown that *a solution exists if and only if there is a vector field* $w(x)$ *in* $\overline{\Omega}$, *with* $\operatorname{div} w = \frac{\Sigma}{\Omega}$ *in* Ω, $\nu \cdot w = 1$ *on* Σ, *and* $|w| < \frac{1}{\cos\gamma}$ *in* Ω. For certain figures, a field w can be constructed explicitly. For example, for the parallelogram of fig. 6 with coordinate origin at the point of symmetry, the field $w = (u,v)$ with

(12) $$u = \frac{1}{a \sin 2\alpha}\left[x + \left(\frac{1}{b} - \frac{1}{a}\right) y \cot 2\alpha\right]$$

$$v = \frac{1}{b \sin 2\alpha} \; y$$

has the indicated properties whenever $\alpha + \gamma \geq \pi/2$, and thus a solution exists in that case. Since, as we have shown, there can be no solution when $\alpha + \gamma < \pi/2$, the result is sharp.

It is tempting to seek a corresponding result for general polygonal figures; consider however the trapezoidal figure of fig 7. According to the above result, if $a = b$ (that is, for any rectangle) there is a solution whenever $\gamma \geq \pi/4$. However it can be shown that *for any* $\gamma < \pi/2$ *and any* $\varepsilon > 0$, *there is a trapezoid with* $|\alpha - \frac{\pi}{4}| < \varepsilon$, $|\frac{b}{a} - 1| < \varepsilon$, *in which the problem* (5), (7) *has no solution.* Thus, the criterion for a rectangle (or parallelogram) does not apply to a trapezoid. If $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$ a new kind of discontinuous (or at least unstable) dependence seems to appear; an arbitrarily small deviation from the rectangular configuration, throughout which the condition $\alpha + \gamma > \pi/2$ holds uniformly, can lead from existence to nonexistence of a solution.

3. In seeking general conditions for existence of solutions, we may try to minimize $\varphi(\Gamma)$. A comparison with the energy expression (1) shows that there is an exact analogy, in which S, β and 2H are replaced, respectively, by Γ, $\cos\gamma$ and $\frac{\Sigma}{\Omega}\cos\gamma$. Thus we are confronted with the same type of variational problem as the original one, the only differences being that it is now a problem in one lower dimension, in which the container has a general, rather than cylindrical, form. We obtain immediately that *any minimizing curve* Γ *must be a curve of constant mean curvature* $H = \frac{\Sigma}{\Omega}\cos\gamma$ *and hence a circular arc of radius* $R_\gamma = \frac{\Omega}{\Sigma\cos\gamma}$ *, and that* Γ *must meet* Σ *in equal angles* γ*, as measured interior to* Ω^*.

Such a curve Γ may or may not exist, and we must examine the two cases.

<u>Case 1</u> : There is no curve Γ satisfying the indicated conditions . Consider a minimizing sequence of pairs of points $\{p_j, q_j\}$ on Σ, and corresponding $\Gamma_j \subset \Omega$. Since Σ^* is bounded by Σ, the Γ_j are bounded in length, hence admit a convergent subsequence for which p_j, q_j tend to points p, q (not necessarily distinct) on Σ. The limit curve Γ must coincide with Σ^*, since if it contained a subarc in Ω it would, by hypothesis, not satisfy the necessary conditions and hence not provide a minimum. Thus $\Omega^* = \emptyset$, $\varphi(\Gamma) = (1 - \cos\gamma)\Sigma^* \geq 0$, hence (10) holds for every $\Gamma \subset \Omega$ and by Giusti's theorem a solution exists.

Thus, *the nonexistence of an extremal curve* Γ *that meets* Σ *in equal angles* γ *is a sufficient condition for existence of a solution of* (5), (7) *in* Ω.

<u>Case 2</u> : There exists an extremal Γ with the indicated properties. We examine the configuration from the viewpoint of the classical calculus of variations and ask whether Γ can be embedded in a field.

<u>Lemma</u> : *Any two diametrically opposite points on an extremal curve* Γ *are conjugate to each other in the sense of Jacobi. These points are conjugate to no other points on* Γ.

Hence, *if the arc* Γ *under consideration is a subarc of a semicircle, a field embedding is possible and it follows that* Γ *provides a strict relative minimum for the functional* φ. *If* Γ *strictly includes a semicircle, it will not provide even a local minimum.*

We may thus restrict attention to extremal arcs Γ that are subarcs of semicircles $\left(\text{of radius } R_\gamma = \frac{\Omega}{\Sigma \cos\gamma}\right)$ and which meet Σ in equal angles γ (measured within Ω^*). Every such arc provides a strict relative minimum φ_m for φ; in order to obtain information with regard to existence of solutions we must determine whether φ_m is positive or not. As an example, we recall the "tail domain" of Chen (for $\gamma = 0$). Chen was able to show that in every such configuration, $\varphi_m < 0$, so that no solution will exist.

4. A configuration appropriate to the present considerations is provided by the trapezoid, of which we illustrate one side adjacent to the line of symmetry in fig.8. We restrict attention to values $\gamma \geq \frac{\pi}{2} - \alpha$, since otherwise the corner condition will exclude existence. We suppose also, for reasons of technical implicity, that $\alpha > \pi/6$. The following results can be demonstrated. For simplicity, we have normalized a=

i) Given l, γ, with $\frac{\pi}{2} - \alpha \leq \gamma < 2\alpha$, there is a unique circular arc Γ of radius R_γ, meeting Σ in equal angles γ. Γ appears as in the figure, although it need not always lie interior to the trapezoid. (If $\gamma \geq 2\alpha$, no such Γ exists, and thus the problem (5), (7) has a solution).

ii) There holds $\tau > \delta$. *There holds* $\delta > 0$ *if and only if both inequalities*

(13)
$$\left.\begin{aligned} &\cos\gamma > \frac{l(2 + l\cos 2\alpha)}{(l-2)\,l\cos 2\alpha - 4}\cos 2\alpha \\ &(l - 2)\,l\cos 2\alpha - 4 > 0 \end{aligned}\right\}$$

are satisfied. If $\delta \leq 0$ a solution exists. If $\delta > 0$, Γ provides a local minimum φ_m for $\varphi(\Gamma)$. A solution of (5), (7) exists for the given Γ if and only if $\varphi_m > 0$.

iii) *For fixed* α *and all sufficiently large* l, *there is a unique* $\gamma = \gamma_{cr}$ *for which* (13) *holds and for which* $\varphi_m = 0$. *A solution of* (5), (7) *exists if and* *only if* $\gamma > \gamma_{cr}$.

iv) For fixed α, let l increase. Then γ_{cr} and the corresponding δ_{cr} both increase.

v) *For fixed* α, *let* $l \to \infty$. *Then* $\gamma_{cr} \nearrow 2\alpha$, *and setting*

$$\sigma_{cr} = 2\alpha - \gamma_{cr},$$

(14) $$l\,\sigma_{cr}^2 \to 2 \sin 4\alpha$$

(15) $$\frac{1}{\sqrt{l}}\,\delta cr \to \frac{2 \sin^2\alpha}{\sqrt{1-2\sin^2\alpha}}$$

vi) For fixed α, fix l large enough that $\delta_{cr} > 0$. Let γ_j be a decreasing sequence such that $\gamma_j \searrow \gamma_{cr}$. For each γ_j there is a solution $u^{(j)}$ of (5), (7), unique up to an additive constant. We normalize the solutions to vanish at a fixed point of $\Omega \setminus \Omega^*$. *Then (uniformly in compacta)* $u^{(j)} \to U$ *a solution of* (7) *in* $\Omega \setminus \Omega^*$, $u^{(j)} \to \infty$ *in* Ω^*. *On* Γ_{cr} *there holds* $|\nabla u^{(j)}| \to \infty$. *The solution surface defined by* U *in* $\Omega \setminus \Omega^*$ *is asymptotic to a vertical circular cylinder over* Γ_{cr}, *as* Γ_{cr} *is approached from within* $\Omega \setminus \Omega^*$.

5. We note that the singular behaviour in the trapezoid differs from that which occurs at the critical γ in a corner. In the trapezoid the solution exists for a half open interval $\gamma_{cr} < \gamma \le \pi/2$, and becomes singular as $\gamma \searrow \gamma_{cr}$. In a corner of, e.g., a regular polygon, the solution exists for a closed interval $\frac{\pi}{2} - \alpha \le \gamma \le \frac{\pi}{2}$, then disappears discontinuously for smaller γ. We note also that at a corner, if $\alpha + \gamma \ge \frac{\pi}{2}$ there is no minimizing curve Γ, whereas if $\alpha + \gamma < \frac{\pi}{2}$ a minimizing curve exists and yields always $\varphi(\Gamma) < 0$ (thus excluding existence). For the trapezoid, minimizing curves Γ can occur for which $\varphi(\Gamma) > 0$ (see iii) of §4 above).

6. The trapezoid provides an example to show that *a capillary surface over a convex domain in the absence of gravity need not be convex*. See Korevaar [5] where an example is given for the case in which the gravity field does not vanish.

7. From the point of view of general existence criteria, the domain Ω^* of fig 8 provides a formal analogue, for $\gamma > 0$, of the tail domain of Chen (see 2. above). However if $\gamma > \gamma_0$ a solution exists, and we thus see that the result of Chen does not extend to the case $\gamma \neq 0$. General geometrical

necessary conditions, beyond those given by Theorem 3 of [1], remain to be formulated.

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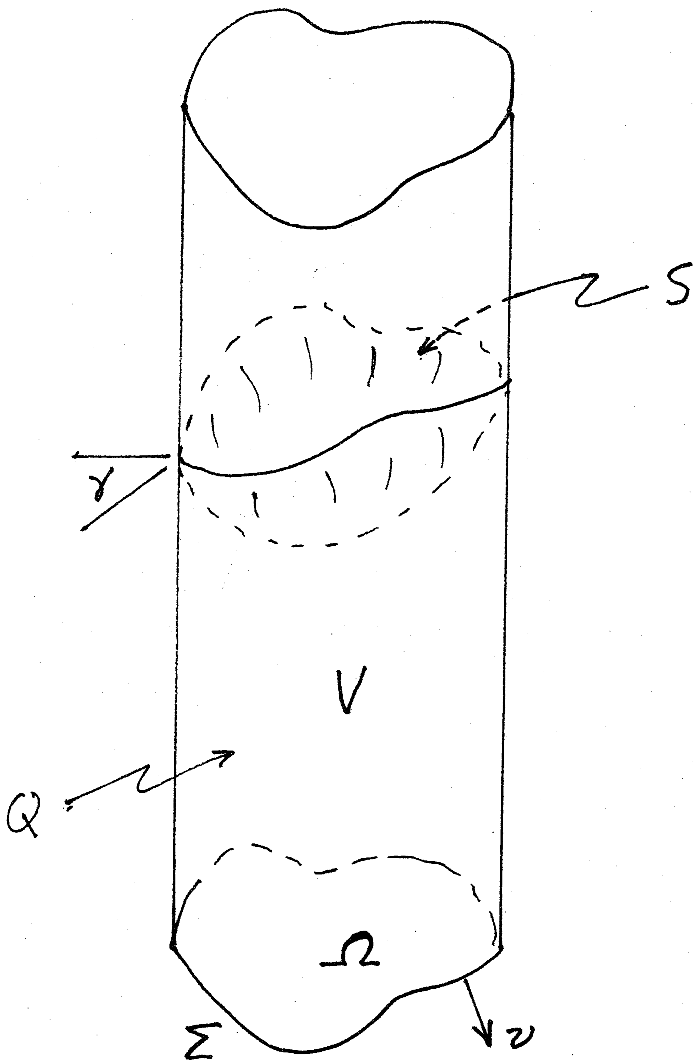

Figure 1

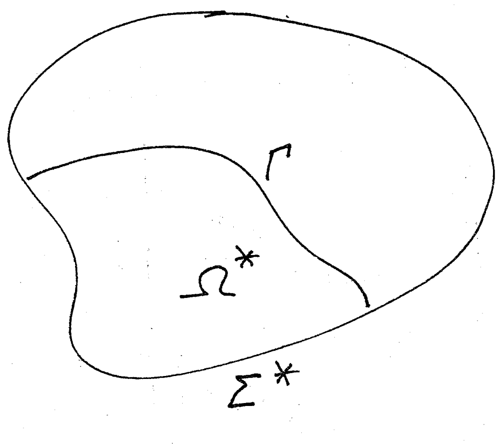

Figure 2

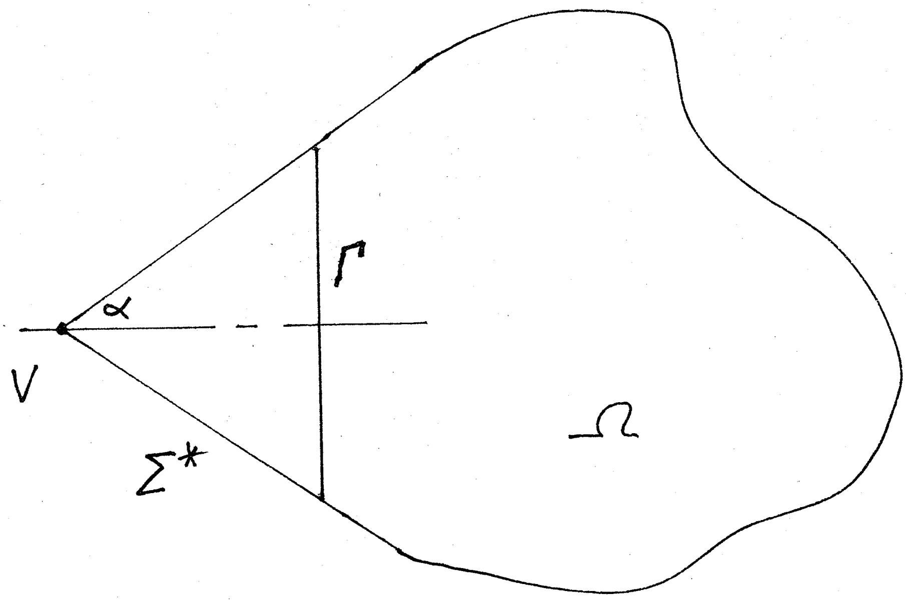

Figure 3

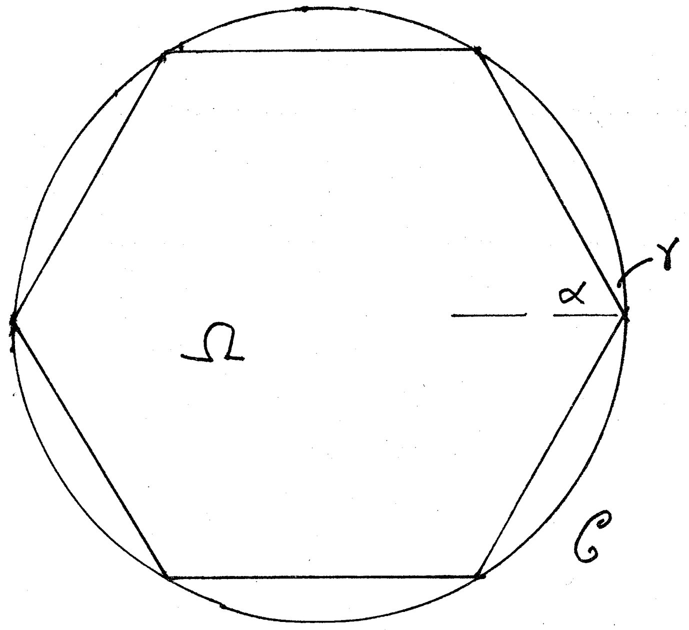

Figure 4

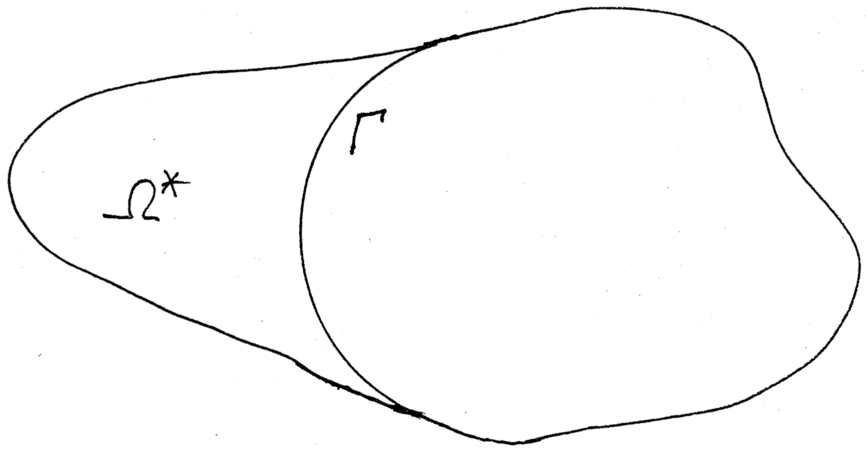

Figure 5

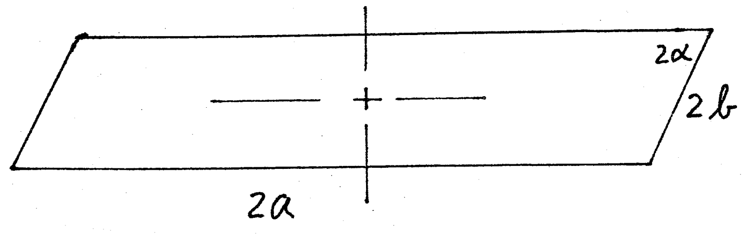

Figure 6

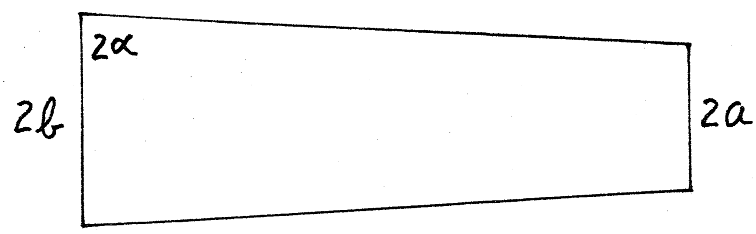

Figure 7

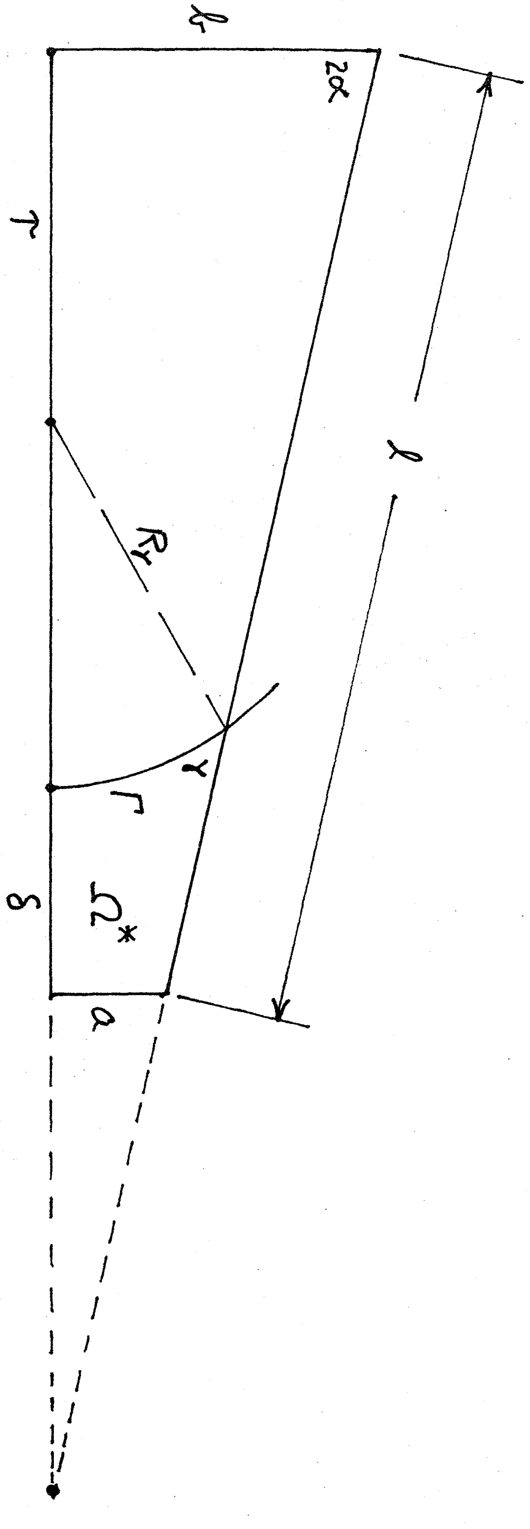

Figure 8