数理解析研究所講究録
第 432 巻 1981 年 1-12

NONLINEAR OSCILLATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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1. INTRODUCTION

In this talk we consider the nonlinear second order functional differential equation with complicated deviating argument

(1) $$[r(t)y'(t)]' + f(y(h(t)),t) = 0.$$

The conditions we always assume for r, g, f are as follows:

(a) $r(t)$ is continuous and positive for $t \geq \alpha$;

(b) $h(t) = g(t) + k(y(t))^2$, where k is a nonnegative constant , $g(t)$ is continuous for $t \geq \alpha$ and $g(t) \geq t$;

(c) $f(y,t)$ is continuous for $|y| < \infty$, $t \geq \alpha$, and $yf(y,t) > 0$ for $y \neq 0$, $t \geq \alpha$.

Equation (1) is called superlinear if, for each fixed t, $f(y,t)/y$ is nondecreasing in y for $y > 0$ and nonincreasing in y for $y < 0$. It is called strongly superlinear if there exists a number $\sigma > 1$ such that, for each fixed t, $f(y,t)/|y|^{\sigma}\operatorname{sgn} y$ is nondecreasing in y for $y > 0$ and nonincreasing in y for $y < 0$. Equation (1) is called sublinear if, for each fixed t, $f(y,t)/y$

is nonincreasing in y for $y > 0$ and nondecreasing in y for $y < 0$. It is called strongly sublinear if there exists a number $\tau < 1$ such that, for each t, $f(y,t)/|y|^{\tau}\operatorname{sgn} y$ is nonincreasing in y for $y > 0$ and nondecreasing in y for $y < 0$. In what follows we restrict our discussion to those solutions $y(t)$ of (1) which exist on some ray $[T_y,\infty)$ and satisfy $\sup\{|y(t)|:t \geq T\} > 0$ for every $T > T_y$. Such a solution is said to be oscillatory if the set of its zeros is not bounded; otherwise, it is said to be nonoscillatory. Equation (1) itself is called oscillatory if all of its solutions are oscillatory. An important special case of (1) is the following generalized Emden-Fowler equation

$$[r(t)y'(t)]' + p(t)|y(h(t))|^{\gamma}\operatorname{sgn} y(h(t)) = 0 \tag{2}$$

where γ is a positive constant and $p(t)$ is a continuous and nonnegative function on $[\alpha,\infty)$. The problem of oscillation of solutions of functional differential equations with deviating arguments has received a wide attention during the last several years. Most of the literature, however, has been devoted to the investigation of differential equations with retarded arguments, and little is known about the oscillatory behavior of differential equations with complicated deviating arguments.

The main purpose of this paper is to undertake a first attempt in the direction of establishing oscillation and non-oscillation results of all solutions for equation (1) with complicated argument which seem to be interesting in the engineering. To do this, we refere to the papers[1-5,7], Hino[12] and the book([6], p. 3). Especially, we get a hint for this problem from an equation

$$\frac{du(t)}{dt} = u(t)(1-u(t-h(t,u(t))))$$

in the book([6], p. 3).

2. THE MAIN RESULTS

In what follows we use the function R(t) defined by

$$R(t) = \int_{\alpha}^{t} r(s)^{-1}ds, \quad \text{and} \quad \lim_{t\to\infty} R(t) = \infty.$$

THEOREM 1. *Let* (1) *be either superlinear or sublinear. Then, a necessary and sufficient condition for* (1) *to have a bounded nonoscillatory solution is that*

$$(3) \qquad \int^{\infty} R(t)|f(c,t)|dt < \infty \quad \textit{for some} \quad c \neq 0.$$

PROOF. (*Necessity*) Let y(t) be a bounded nonoscillatory solution of (1). Multiplying (1) by R(t) and integrating from t_1 to t, we have

$$R(t)r(t)y'(t) - y(t) - R(t_1)r(t_1)y'(t_1) + y(t_1) + \int_{t_1}^{t} R(s)f(y(h(s)),s)ds = 0,$$

which implies (3).

(*Sufficiency*) Consider the integral equation

$$(4)\quad y(t) = a + \int_T^t R(s)f(y(h(s)),s)ds + R(t)\int_t^\infty f(y(h(s)),s)ds.$$

A solution of (4) is a solution of equation (1). By using Schauder's fixed point theorem, we have a solution y(t) of (4) which tends to a finite limit as $t \to \infty$.

THEOREM 2. *Let (1) be strongly superlinear. Then, a necessary and sufficient condition for (1) to be oscillatory is that*

$$(5)\qquad \int^\infty R(t)|f(c,t)|dt = \infty \text{ for all } c \neq 0.$$

THEOREM 3. *Let (1) be strongly sublinear. Suppose that f(y,t) is nondecreasing in y for each t. If*

$$(6)\qquad \int^\infty |f(cR(t),t)|dt = \infty \text{ for all } c \neq 0,$$

then (1) is oscillatory.

EXAMPLE. Consider the functional differential equation with complicated argument

$$[t^{-2}y'(t)]' + 2^{\frac{2}{3}}t^{-\frac{11}{3}}[y(t^2 + y(t)^2)]^{\frac{1}{3}} = 0,$$

Which has a nonoscillatory solution y(t) = t. As is easily seen (6) is violated.

3. THE MORE GENERAL CASE

We consider the case h(t) = g(t) + s(y(t),y'(t)), where

$s(u,v)$ is continuous on R^2 and $s(u,v) \geqq 0$ for all $(u,v) \in R^2$, and $g(t) \geqq t$.

(7) $\quad [r(t)y'(t)]' + f(y(g(t) + s(y(t),y'(t))),t) = 0.$

THEOREM 4. *Let* (7) *be strongly superlinear. Then, a necessary and sufficient condition for* (7) *to be oscillatory is* (5).

4. THE CASE WHERE $\int_{\alpha}^{\infty} r(t)^{-1}dt < \infty$

We use the function $\rho(t)$ defined by $\rho(t) = \int_{t}^{\infty} r(s)^{-1}ds$.

THEOREM 5. *Let* (1) *be strongly superlinear. Suppose that*

(8) $$\int^{\infty} |f(c\rho(t+M),t)|dt = \infty \text{ for all } c \neq 0,$$

and $h(t) = t + s(y(t),y'(t))$, $0 \leq s(u,v) \leq M$ *for all* $(u,v) \in R^2$ *and* M *is a constant. Then* (1) *is oscillatory.*

PROOF. Suppose there exist a nonoscillatory solution $y(t)$ of (1). Without loss of generality we may suppose that $y(t) > 0$ for $t \geqq t_0$. By adding some modifications to the proof part of Kusano and the present speaker([7], pp. 548-549), we have that

$$(\sigma-1)k^{-\sigma} \int_{t_2}^{t} f(k\rho(h(s)),s)ds \leq [-r(t_2)y'(t_2)]^{1-\sigma} - [-r(t)y'(t)]^{1-\sigma},$$

where $k = -r(t_2)y'(t_2)$.

From this, we obtain

$$\int_{t_2}^{\infty} f(k\rho(s+M),s)ds < \int_{t_2}^{\infty} f(k\rho(h(s)),s)ds < \infty,$$

which implies a contradiction to (8).

5. THE HIGHER ORDER CASE

We consider the higher order equation

(9) $$L_n y(t) + H(t,y(h_1(t)),\ldots,y(h_m(t))) = f(t),$$

where $n \geqq 2$, $h_i(t) = g_i(t) + s_i(y(t),y'(t),\ldots,y^{(n-1)}(t))$, $g_i(t) \geqq t$, $1 \leqq i \leqq m$, $s_i(y_1,y_2,\ldots,y_n)$ is continuous on R^n, $s_i(y_1,y_2,\ldots,y_n) \geqq 0$ for any $(y_1,y_2,\ldots,y_n) \in R^n$, and L_n denotes the differential operator

(10) $$L_n = \frac{1}{p_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \frac{\cdot}{p_0(t)}.$$

We assume that p_i, f, g_i: $[a,\infty) \to R$ and $H:[a,\infty) \times R^m \to R$ are continuous and $p_i(t) > 0$. (Cf. Singh and Kusano[9])

A) MAIN RESULT

Consider the case

(11) $$\int_a^{\infty} p_i(t)dt = \infty \qquad \text{for } 1 \leqq i \leqq n-1.$$

A differential operator L_n defined by (10) is said to be in canonical form if condition (11) is satisfied. It is shown that any differential operator of the form (10) can be represented in canonical form in an essentially unique manner. We will use the following notations:

(12)
$$\begin{cases} I_0 = 1, \\ I_k(t,s,;p_{i_k},\ldots,p_{i_1}) = \int_s^t p_{i_k}(r) I_{k-1}(r,s;p_{i_{k-1}},\ldots,p_{i_1})dr \end{cases}$$

and

(13) $J_i(t,s) = p_0(t)I_i(t,s;p_1,\ldots,p_i)$, $J_i(t) = J_i(t,a)$,

(14) $K_i(t,s) = p_n(t)I_i(t,s;p_{n-1},\ldots,p_{n-i})$, $K_i(t) = K_i(t,a)$,

for $i_k \in \{1,2,\ldots,n-1\}$, $1 \leq k \leq n-1$, and $t, s \in [a,\infty)$.

THEOREM 6. *Suppose* (11) *holds and there exists a continuous function* $q: [a,\infty) \to [0,\infty)$ *such that*

(15) $|H(t,y_1,\ldots,y_m)| \leq q(t)$ *for* $(t,y_1,\ldots,y_m) \in [a,\infty)\times R^m$

holds. Suppose that

$$\int^\infty K_{n-1}(t)q(t)dt < \infty \quad \text{and} \quad \int^\infty K_{n-1}(t)|f(t)|dt < \infty.$$

Then every oscillatory solution $y(t)$ *of* (8) *satisfies*

$$\lim_{t\to\infty} [y(t)/p_0(t)] = 0.$$

B) NON CANONICAL L_n

We consider the case where L_n in (9) is not in canonical form. Any differential operator of the form (10) can be represented in canonical form and the representation is essentially unique. More precisely, if L_n is given by (10) and if condition (11) is not satisfied, then L_n can be rewritten as

$$(16) \qquad L_n = \frac{1}{\tilde{p}_n(t)} \frac{d}{dt} \frac{1}{\tilde{p}_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{\tilde{p}_1(t)} \frac{d}{dt} \frac{\cdot}{\tilde{p}_0(t)},$$

so that

$$(17) \qquad \int_a^\infty \tilde{p}_i(t)dt = \infty, \quad 1 \leqq i \leqq n-1,$$

and the $\tilde{p}_i(t)$, $o \leqq i \leqq n$, are determined up to positive multiplicative constants with product 1. By a principal system for L_n is meant a set of n solutions $Y_1(t),\ldots,Y_n(t)$ of $L_n y(t) = 0$ which are eventually positive and satisfy

$$(18) \qquad \lim_{t\to\infty} \frac{Y_i(t)}{Y_j(t)} = 0 \quad \text{for } 1 \leqq i < j \leqq n.$$

In case L_n is in canonical form the set of functions

$$(19) \qquad \{J_0(t),\ J_1(t),\ldots,J_{n-1}(t)\}$$

defined by (13) is a principal system for L_n, and the set of functions

$$(20) \qquad \{K_0(t),\ K_1(t),\ldots,K_{n-1}(t)\}$$

defined by (14) is a principal system for the operator

$$(21)\quad M_n = \frac{1}{p_0(t)} \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \frac{\cdot}{p_n(t)} ,$$

which is also in canonical form. For a general operator L_n a principal system can easily be obtained by direct integration of the equation $L_n y(t) = 0$. A basic property of principal system is that if $\{Y_1(t),\dots,Y_n(t)\}$ and $\{\tilde{Y}_1(t),\dots,\tilde{Y}_n(t)\}$ are any two principal systems for the same L_n, then the limits

$$(22)\qquad \lim_{t\to\infty} \frac{\tilde{Y}_i(t)}{Y_i(t)} > 0,\ 1 \leq i \leq n,$$

exist and are finite.

THEOREM 7. *Suppose that there exists a continuous function* $q:[a,\infty)\to R$ *such that*

$$|H(t,y_1,\dots,y_m)| \leq q(t) \text{ for } (t,y_1,\dots,y_m) \in [a,\infty) \times R^m,$$

holds. Let $\{Y_1(t),\dots,Y_n(t)\}$ *be a principal system for* L_n *and let* $\{Z_1(t),\dots,Z_n(t)\}$ *be a principal system for* M_n *defined by* (21). *Suppose that*

$$\int^\infty Z_1(t)q(t)dt < \infty \text{ and } \int^\infty Z_1(t)|f(t)|dt < \infty$$

hold, then every proper solution $y(t)$ *of* (9) *satisfies*

$$y(t) = O(Y_n(t)) \text{ as } t \to \infty.$$

C) NONOSCILLATION THEOREM

We consider the equation

(23) $$L_n y(t) + a(t)Q(y(h_1(t)),\dots,y(h_m(t))) = 0,$$

where a, h_j: $[a,\infty) \to R$, $1 \leq j \leq m$, and Q: $R^m \to R$ are continuous.

THEOREM 8. Suppose that (11) holds, $Q(y_1,\dots,y_m)$ is bounded and

$$Q(y_1,\dots,y_m) = 0(|y_1|) \text{ as } y_1 \to 0.$$

If in addition

$$\liminf_{t\to\infty} \frac{1}{p_0(t)} > 0 \text{ and } \int^{\infty} K_{n-1}(t)|a(t)|dt < \infty,$$

then all proper solutions of (23) are nonoscillatory.

REMARK. This contain the result of [11].

** A part of this talk shall be published in [8].

REFERENCES

[1] J. B. Bradley, Oscillation theorems for a second order delay equation, J. Differential Equations 8(1970), 397-403.

[2] Ya. V. Bykov, L. Ya. Bykova and E. I. Ševcov, Sufficient conditions for the oscillation of solutions of nonlinear differential equations with deviating argument, Differential'nye Uravnenija 9(1973), 1555-1560. (Russian)

[3] Ya. V. Bykov and G. D. Merzlyakova, On the oscillation of solutions of nonlinear differential equations with deviating argument, Differencial'nye Uravnenija 10(1974), 210-220. (Russian)

[4] K. L. Chiou, Oscillation and nonoscillation theorems for second order functional differential equations, J. Math. Anal. Appl. 45(1974), 382-403.

[5] C. V. Coffman and J. S. W. Wong, Oscillation and nonoscillation theorems for second order ordinary differential equations, Funkcial. Ekvac. 15(1972), 119-130.

[6] J. Kato, Functional equation, Chikumashobo(1974). (Japanese)

[7] T. Kusano and H. Onose, Nonlinear oscillation of second order functional differential equations with advanced argument, J. Math. Soc. Japan 29(1977), 541-559.

[8] Hiroshi Onose, Nonlinear oscillation of functional differential equations with complicated deviating argument, Bull. Fac. Sci ., Ibaraki Univ., Math.. (to appear)

[9] B. Singh and T. Kusano, Asymptotic behavior of oscillatory solutions of a differential equation with deviating arguments. (to appear)

[10] B. Singh, Asymptotically vanishing oscillatory trajectories in second order retarded equations, SIAM J. Math. Anal. 7(1976), 37-44.

[11] V. A. Staikos and Ch. G. Philos, Nonoscillatory phenomena and damped oscillations, Nonlinear Anal. 2(1978), 197-210.

[12] Yoshiyuki Hino, On oscillation of the solution of second order functional differential equations, Funkcial. Ekvac. 17(1974), 95-105.

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