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Total Stability and Uniform Asymptotic Stability for Delay Differential Equations

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For ordinary differential equations and functional differential equations, there are many results concerning with relationships between the total stability and the uniform asymptotic stability (cf. [1, 5, 6, 8, 9]).

For ordinary differential equations, Gorsin [1] and Malkin [5] proved that, under fairly general assumptions, uniform asymptotic stability implies the total stability. It is known that the converse is not generally true (cf. [6]). However, Massera[6] proved that the null solution of linear homogeneous system is uniformm asymptotic stability, if it is total stable. Furthermore, Seifert [8] has extended Massera's result to the more general systems.

Massera's theorem. If the null solution of the linear system

$$(*) \qquad \dot{x} = A(t)x, \quad x \in R^n,$$

where $A(t)$ is $n \times n$ matrix and continuous on I, $I = [0, \infty)$, is totally stable, then it is uniformly asymptotically stable.

Definition 1. The null solution of (*) is said to be totally stable, if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $g(t, x)$ is continuous and satisfies $|g(t, x)| < \delta(\varepsilon)$ on $[s, \infty) \times R^n$, $|x| \leqq \varepsilon$, for an $s \geqq 0$, and if $|x^0| < \delta(\varepsilon)$, then the solution of $\dot{x} = A(t)x + g(t, x)$ through (s, x^0) satisfies $|x(t)| < \varepsilon$ for all $t \geqq s$, where $|x|$ is any norm of $x \in R^n$.

Proof of Massera's theorem. If the null solution of (*) is totally stable, there exists a $\delta > 0$ such that if $|y^0| < \delta$, the solution $y(t, s, y^0)$ of $\dot{y} = A(t)y + \delta y$, where $|y| < 1$, satisfies $|y(t, s, y^0)| < 1$. But the solutions of both equations are related by $y(t, s, y^0) = x(t, s, y^0)e^{\delta(t - s)}$, where $x(t, s, y^0)$ is the solution of (*) through (s, y^0), which proves theorem.

Suppose $0 < r \leqq \infty$ is given. If $x: [\sigma-r, \sigma+A) \to R^n$, $A > 0$, is a given function, let x_t be defined by $x_t(\theta) = x(t+\theta)$, $-r \leqq \theta \leqq 0$, for each $t \in [\sigma, \sigma+A)$.

Let F_n denote a set of R^n-valued functions on $[-r, 0]$ or $(-\infty, 0]$. For linear functional differential equations

$$(**) \qquad \dot{x} = A(t, x_t),$$

where $A(t, \phi)$ is continuous on $I \times F_n$ and linear in $\phi \in F_n$, let $x(t, s, \phi^0)$ be the solution of (**) through (s, ϕ^0). Then $y(t) = x(t, s, \phi^0)e^{\delta(t - s)}$, $s \in I$, satisfies

$$\dot{y}(t) = \dot{x}(t, s, \phi^0)e^{\delta(t - s)} + \delta x(t, s, \phi^0)e^{\delta(t - s)}$$

$$= A(t, x_t(s, \phi^0)e^{\delta(t - s)} + \delta y(t)$$

$$= A(t, e^{\delta(t - s)} \times y(t + \cdot)e^{-\delta(t + \cdot - s)}) + \delta y(t)$$

$$= A(t, y(t + \cdot) \times e^{-\delta \cdot}) + \delta y(t)$$

$$= A(t, \tilde{y}_t) + \delta y(t),$$

where $\tilde{\phi} = \{\phi(\cdot) \times e^{-\delta \cdot} | \phi \in F_n\}$.

If $F_n = C([-r, 0], R^n)$, then, clearly, $\phi \in C$ implies $\tilde{\phi} \in C$ for any $\delta > 0$. However, if $F_n = C_\gamma$, $C\gamma = \{\phi | \text{continuous on } (-\infty, 0], \phi(\theta)e^{\gamma\theta} \to \text{exists as } \theta \to -\infty, \gamma > 0\}$, $\phi \in C_\gamma$ does not imply $\tilde{\phi} \in C_\gamma$ for some $\phi \in C_\gamma$. Hence if $r = \infty$, we can not apply Massera's idea for functional differential equations with infinite delay to obtain the same result as that of Massera's.

We shall give the space B discussed by Hale and Kato [2]. Let B be a real linear vector space of functions mapping $(-\infty, 0]$ into R^n with a semi-norm $|\cdot|_B$. For any elements ϕ and ψ in B, $\phi = \psi$ means $\phi(t) = \psi(t)$ for all $t \in (-\infty, 0]$. The space B is assumed to have the following properties:

(I) If $x(t)$ is defined on $(-\infty, a)$, continuous on $[\sigma, a)$, $\sigma < a$, and $x_\sigma \in B$, then for $t \in [\sigma, a)$, (I.1) $x_t \in B$,

(I.2) x_t is continuous in t with respect to $|\cdot|_B$,

(I.3) there are a $K > 0$ and a positive continuous function $M(\beta)$, $M(\beta) \to 0$ as $\beta \to \infty$, such that $|x_t|_B \leqq K \sup_{\sigma \leqq s \leqq t} |x(s)| + M(t-\sigma)|x_\sigma|_B$.

(II) $|\phi(0)| \leqq M_1|\phi|_B$ for $M_1 > 0$.

Consider the system

$$\dot{x}(t) = A(t, x_t), \tag{1}$$

where $A(t, \phi)$ is continuous in $(t, \phi) \in I \times B$ and linear in ϕ. Then we have $|A(t, \phi)| \leqq L(t)|\phi|_B$ for a continuous function $L(t)$ on I. Assume that $L(t)$ is a constant, that is, there exists an $L > 0$ such that

$$|A(t, \phi)| \leqq L|\phi|_B \text{ on } I \times B. \tag{A}$$

Remark. It is known that if $A(t, \phi)$ is almost periodic in t uniformly for $\phi \in \overline{B_H}$, $\overline{B_H} = \{\phi \in B;\ |\phi|_B \leqq H\}$, then Condition (A) holds good.

We shall give some definitions of stabilities.

Definition 2. The null solution of (1) is said to be totally stable, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $g(t, \phi)$ is continuous and satisfies $|g(t, \phi)| < \delta(\varepsilon)$ on $[s, \infty) \times B_\varepsilon$, for an $s \geqq 0$ and if $|\phi^0|_B < \delta(\varepsilon)$, then the solution $x(t)$ of

(2) $$\dot{x}(t) = A(t, x_t) + g(t, x_t)$$

through (s, ϕ^0) satisfies $|x(t)| < \varepsilon$ for all $t \geqq s$.

Definition 3. The null solution of (1) is said to be uniformly stable, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $s \geqq 0$ and $|\phi^0|_B < \delta(\varepsilon)$, then the solution $x(t)$ of (1) through (s, ϕ^0) satisfies $|x(t)| < \varepsilon$ for all $t \geqq s$.

Definition 4. The null solution of (1) is said to be uniformly asymptotically stable, if it is uniformly stable and if for any $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$ such that the solution $x(t)$ of (1) through (s, ϕ^0), $s \geqq 0$, $|\phi^0|_B < 1$, satisfies $|x(t)| < \varepsilon$ for all $t \geqq s + T(\varepsilon)$.

Remark. These concepts of stabilities obviously require that $|\phi(0)| < \varepsilon$ if $|\phi|_B < \delta(\varepsilon)$. This property is guaranteed by Property (II).

Theorem. Suppose that the space B has properties (I) and (II) and $A(t, \phi)$ satisfies the condition (A). Then, the null solution of (1) is totally stable if and only if it is uniformly asymptotically stable.

We shall use the following lemma to prove the necessity part in Theorem. It can be proved by using the standard arguments. For details, see [3].

Lemma. The null solution of (1) is totally stable if and only if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $g(t, \phi)$ is continuous on $[s, \infty) \times B$, $s \in I$, and satisfies $|g(t, x_t)| < \delta(\varepsilon)$ as long as $x(t)$ is continuous on $[s, t]$, $|x(t)| \leqq \varepsilon$ for $t \geqq s$ and $|x_s|_B < \delta(\varepsilon)$, then any solution $x(t)$ of

$$\dot{x}(t) = A(t, x_t) + g(t, x_t)$$

through (s, x_s) satisfies $|x(t)| < \varepsilon$ for all $t \geqq s$, as long as it exists.

Proof of theorem. First, we shall show the sufficiency of the condition. It is known by Sawano[7] that if the null solution of (1) is uniformly asymptotically stable, then there exists a continuous real-valued functional $V(t, \phi)$ defined on $I \times B$ which satisfies the following conditions:

(a.1) $|\phi|_B \leqq V(t, \phi) \leqq N|\phi|_B$, where N is a constant.

(a.2) $|V(t, \phi^1) - V(t, \phi^2)| \leqq N|\phi^1 - \phi^2|_B$.

(a.3) $V_{(1)}(t, \phi) = \limsup_{\delta \to 0^+} \{V(t+\delta, x_{t+\delta}) - V(t, \phi)\}/\delta \leqq -cV(t, \phi)$, where $x(t)$ is the solution of (1) through (t, ϕ) and c is a positive constant.

(The phase space considered by Sawano[7] is slightly different from ours. however, it is not difficult to see that the hypotheses in Sawano's paper play their role almost only through the relation (I.3). Furthermore, it is known by Kato[4] that the concept of uniform asymptotic stability given by Sawano is equivalent to ours under the properties (I) and (II).)

Hence, by using same arguments as in the proof for ordinary differential equations, the sufficiency of the condition can be proved.

(Clearly, in the above proof, we can drop Condition (A).)

Next, we shall show the necessity of the condition. We note that the solutions of $\dot{x}(t) = A(t+s, x_t)$ behave in the same manner as those of (1) for any fixed $s \geqq 0$, because the uniformities of L in (A) and of the total stability play the essential roles in the proof and the shape of $A(t, \phi)$ itself has no direct effect as we will see in the below. Therefore it will be enough to show that for any $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$ such that the solution $x(t)$ of (1) through $(0, \phi^0)$, $|\phi^0|_B < \delta(1)$, satisfies $|x(t)| < \varepsilon$ for all $t \geqq T(\varepsilon)$, where $\delta(\cdot)$ is the one given for the total stability of the null solution of (1).

Let $x(t)$ be a solution of (1) satisfying $|x_0|_B < \delta(1)$, and hence $|x(t)| < 1$ for all $t \geqq 0$. The idea is that we find a positive scalar continuous function $u(t, \varepsilon)$, for a given $\varepsilon > 0$, defined on $(-\infty, \infty)$ such that

(b.1) $u(t, \varepsilon) = 1$ for $t \leqq 0$ and $u(t, \varepsilon) \in C^1$ on $(0, \infty)$,

(b.2) there exists a $T(\varepsilon) > 0$ such that $u(t, \varepsilon) > 1/\varepsilon$ for

all $t \geqq T(\varepsilon)$,
for which the function

$$(2) \qquad y(t) = u(t, \varepsilon)x(t)$$

satisfies $|y(t)| \geqq 1$ for all $t \geqq 0$, and then the proof will be completed, because $|x(t)| = |y(t)/u(t, \varepsilon)| < \varepsilon$ for all $t \geqq T(\varepsilon)$ by (b.2).

We note that the function $y(t)$ given by (2) satisfies $|y_0|_B = |x_0|_B < \delta(1)$ by (b.1) and it is a solution of

$$(3) \qquad y(t) = A(t, y_t) + g(t, y_t)$$

with $g(t, y_t) = \dot{u}(t, \varepsilon)y(t)/u(t, \varepsilon) + A(t, u(t, \varepsilon)x_t) - A(t, y_t)$. Therefore, in order to show that the solution $y(t)$ of (3) given by (2) satisfies $|y(t)| \leqq 1$ for all $t \geqq 0$, it is sufficient to find a condition for $u(t, \varepsilon)$ to guarantee that $|g(t, y_t)| < \delta(1)$ as long as $|y(t)| \leqq 1$ by Lemma, because the null solution of (1) is totally stable. The following conditions together with (b.1) and (b.2) will be sufficient for our purpose:

(b.3) $|\dot{u}(t, \varepsilon)/u(t, \varepsilon)| < \delta(1)/3$ for all $t \geqq 0$,

(b.4) $u(t, \varepsilon) < 2/\varepsilon$ for all $t \geqq 0$,

(b.5) $|u(t, \varepsilon) - u(s, \varepsilon)| < \min\{\delta(1)/3LK, 1/3LM\}$, if $|t - s| \leqq S(\varepsilon)$, where $S(\varepsilon)$ is chosen so that $LM(S(\varepsilon))(1 + (2/\varepsilon))(K + M(1)) < \delta(1)/3$ and $M = \sup_{\beta \geqq 0} M(\beta)$.

In fact, if $|y(s)| \leqq 1$ for all $s \leqq t$, we have $|\dot{u}(t, \varepsilon)y(t)/u(t, \varepsilon)| \leqq \delta(1)/3$ by (b.3) and $|A(t, u(t, \varepsilon)x_t) - A(t, y_t)| \leqq L|u(t, \varepsilon)x_t - y_t|_B \leqq L\{K \sup_{\tau\leqq s\leqq t} |u(t, \varepsilon)x(s) - y(s)| + M(t - \tau)|u(t, \varepsilon)x_\tau - y_\tau|_B\} \leqq L\{K \sup_{\tau\leqq s\leqq t} |u(t, \varepsilon) - u(s, \varepsilon)| + M(t - \tau)|u(t, \varepsilon)x_\tau - y_\tau|_B$, where $\tau = \max\{t - S(\varepsilon), 0\}$. Since $|x_\tau|_B \leqq K \sup_{0\leqq s\leqq \tau} |x(s)| + M|x_0|_B \leqq K + M\delta(1)$ and similarly $|y_\tau|_B \leqq K + M\delta(1)$, it follows from (b.4) that $M(t - \tau)|u(t, \varepsilon)x_\tau - y_\tau|_B \leqq M(S(\varepsilon))(1 + (2/\varepsilon))(K + M\delta(1)) \leqq \delta(1)/3$ if $\tau > 0$, while $M(t - \tau)|u(t, \varepsilon)x_\tau - y_\tau|_B \leqq M|u(t, \varepsilon) - 1|\delta(1)$ if $\tau = 0$. Thus we have $|A(t, u(t, \varepsilon)x_t) - A(t, y_t)| \leqq 2\delta(1)/3$ by (b.5), that is, $|g(t, y_t)| = |\dot{u}(t, \varepsilon)y(t)/u(t, \varepsilon) + A(t, u(t, \varepsilon)x_t) - A(t, y_t)| \leqq |\dot{u}(t, \varepsilon)y(t)/u(t, \varepsilon)| + |A(t, u(t, \varepsilon)x_t) - A(t, y_t)| < \delta(1)/3 + 2\delta(1)/3 = \delta(1)$ as long as $|y(s)| \leqq 1$ for all $s \leqq t$.

Thus it is only left to show the existence of a function $u(t, \varepsilon)$ satisfying the required properties (b.1) through (b.5). Such a function will be given by

$$u(t, \varepsilon) = \begin{cases} (1 + 2\alpha t)/(1 + \varepsilon\alpha t), & \text{if } t \geqq 0, \\ 1, & \text{if } t < 0, \end{cases}$$

where we may assume $\varepsilon \in (0, 1)$ and $\alpha = \alpha(\varepsilon)$ is a suitable positive scalar function which satisfies $\alpha(\varepsilon) \leqq \min\{\delta(1)/6, 1/6LMS(\varepsilon), \delta(1)/6LKS(\varepsilon)\}$. Easily we can see that $u(t, \varepsilon)$ is the positive scalar continuous function defined on $(-\infty, \infty)$ and

$u(t, \varepsilon) \in C^1$ on $(0, \infty)$ and $u(t, \varepsilon)$ satisfies all requirements (b.1) through (b.5), since $0 \leqq u(t, \varepsilon) = \alpha(2 - \varepsilon)/(1 + \varepsilon\alpha t)^2 \leqq 2\alpha$, $\dot{u}(t, \varepsilon)/u(t, \varepsilon) = \alpha(2 - \varepsilon)/(1 + \varepsilon\alpha t)(1 + 2\alpha t) \leqq 2\alpha$, for all $t \geqq 0$ and $\lim_{t\to\infty} u(t, \varepsilon) = 2/\varepsilon$.

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