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ON SOME DELAY-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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1. Introduction.

In this paper we consider the linear delay-differential equation involving m delay terms

$$\frac{dx}{dt}(t) = Ax(t) + \sum_{r=1}^{m} A_r x(t-r\tau), \tag{1.1}$$

where τ is a positive constant, $x(t)$ belongs to a Banach space X and A, $A_1, \cdots, A_m$ are linear, not neccessary bounded operators on X. It is assumed that A generates a strongly continuous semi-group $T(t)$ and A_r, $r = 1, \cdots, m$, are relatively bounded compared with A (see section 2).

Our main purpose here is to give a representation of the fundamental solution of (1.1) in terms of $T(t)$ and A_r and establish a variation of constants formula for (1.1). Such expressions are useful to obtain the fundamental theorems and some system theoretical results for (1.1) [9,10]. An application to infinite dimensional linear systems theory is given here.

2. System Description and Mild Solution.

Let us consider the differential system with m delay terms

$$S: \quad \frac{dx}{dt} = Ax(t) + \sum_{r=1}^{m} A_r x(t-r\tau) + f(t), \quad t > 0 \tag{2.1}$$

$$x(0) = x_0, \quad x(s) = g(s), \quad s \in [-m\tau, 0), \tag{2.2}$$

where $\tau > 0$ is a constant, $x(t)$, $f(t)$, $g(s) \in X$ and the operators A and A_r ($r = 1, \cdots, m$), possibly unbounded, are assumed to satisfy the following assumptions H_0 and H_1, respectively.

H_0. A generates a strongly continuous semi-group $\{T(t): t \geq 0\}$ on X

H_1. A_r, $r = 1, \cdots, m$, are closed linear operators with dense domains $D(A_r)$ in X.

First of all we shall give a definition of the mild solution of the system S. To do so we need the next integrability condition for A_r.

H_2^q. For each r there exists a function $M_r(\cdot) \in L_q[0, \tau]$ such that
$\|T(t)A_r x\| \leq M_r(t)\|x\|$ for a.a. $t \in [0, \tau]$ and all $x \in D(A_r)$.

Let the assumption H_2^q, $q \in [1, \infty]$ be satisfied. Then for any $x \in X$, there exists only one element $y(t,x)$ in X for a.a. $t \in [0, \tau]$ as limits of $T(t)A_r x_n$ such that $x_n \in D(A_r)$ and $x_n \to x$ in X. The operator $(T(t)A_r)$ defined by $(T(t)A_r)x = y(t,x)$ for x in X is well-defined and bounded for a.a. $t \in [0, \tau]$. This means that for each r and a.a. $t \in [0, \tau]$ $T(t)A_r$ can be extended to the bounded operator $(T(t)A_r)$ on X and the extended operator $(T(t)A_r)$ satisfies the inequality $\|(T(t)A_r)\| \leq M_r(t)$ for a.a. t in $[0, \tau]$.

Let x_0, $f(\cdot)$ and $g(\cdot)$ be given with

$$x_0 \in X, \tag{2.3}$$

$$f(\cdot) \in L_p^{loc}(R^+; X), \tag{2.4}$$

$$g(\cdot) \in L_{p'}(-m\tau, 0; X), \tag{2.5}$$

and let the assumption $H_2^{q'}$ with $p'^{-1} + q'^{-1} = 1$ be satisfied, where $p, p' \in [1, \infty]$. Then the function

$$x_1(t;\ x_0,f,g) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \sum_{r=1}^{m} \int_0^t (T(t-s)A_r)g(s-r\tau)ds \qquad (2.6)$$

is well-defined, the integrals being Bochner integrals in X, and is strongly continuous on $[0, \tau]$. We give a short proof of this. The first term of (2.6) is clearly strongly continuous by assumption H_0. The integrand of the second term of (2.6) is strongly measurable and Bochner integrable by H_0 and (2.4) and hence again by H_0 the second term is strongly continuous. By assumption $H_2^{q'}$, $(T(t-s)A_r)$ is bounded and its norm is bounded by $M_r(t-s)$, $M_r(\cdot) \in L_{q'}[0, \tau]$, for a.a. $s \in [0, t]$ and hence by (2.5) the function $(T(t-s)A_r)g(s-r\tau)$ is strongly measurable (see Hille-Phillips [7, Chapter 3]) and Bochner integrable on $[0, t]$ (note that $g(s-r\tau) \in D(A_r)$ is not assumed). Therefore all integrands of the third term are Bochner integrable and hence the third term is strongly continuous. Thus $x_1(\cdot;\ x_0,f,g) \in C(0,\tau;\ X)$. In general, for any natural number k we define $x_{k+1}(t;\ x_0,f,g)$ inductively by

$$x_{k+1}(t;\ x_0,f,g) = T(t)x_k(\tau;\ x_0,f,g) + \int_0^t T(t-s)f(k\tau+s)ds$$
$$+ \sum_{r=1}^{m} \int_0^t (T(t-s)A_r) \begin{Bmatrix} x_{k-r+1}(s;\ x_0,f,g), & 1 \le r \le k \\ g(s-(r-k)\tau), & \text{otherwise} \end{Bmatrix} ds \qquad (2.7)$$

Since $x_1 \in C(0,\tau;\ X) \subset L_{p'}(0,\tau;\ X)$ it follows as above that x_2 is well-defined and $x_2 \in C(0,\tau;\ X)$. Continuing this process, we find that $x_k \in C(0,\tau;\ X)$ for all $k = 1,2,\cdots$. Define $x(t;\ x_0,f,g)$ by $x(0;\ x_0,f,g) = x_0$ and $x(t;\ x_0,f,g) = x_k(t-(k-1)\tau;\ x_0,f,g)$ if $t \in ((k-1)\tau, k\tau]$. We shall say that the function $x(\cdot;\ x_0,f,g) \in C(R^+;\ X)$ is the mild solution of S. Since there will be no confusion about the operator $(T(t)A_r)$, we denote this operator simply by $T(t)A_r$.

Under the above assumptions which guarantee the unique existence of the mild solution $x(t;\ x_0,f,g)$, there arises the solution mapping $\mathcal{S} : X \times L_p^{loc}(R^+;\ X) \times L_{p'}(-m\tau,0;\ X) \to C(R^+;\ X)$ defined by $\mathcal{S}(x_0,f,g) = x(\cdot;\ x_0,f,g)$. The domain of $\mathcal{S}$ is the Fréchet space endowed with the product topology of X, $L_p^{loc}(R^+;\ X)$ and $L_{p'}(-m\tau,0;\ X)$. Clearly $C(R^+;\ X)$ is a Fréchet space. By standard but complicated arguments concerning the integral representation of mild solutions given in section 4, we can show that $\mathcal{S}$ is linear and continuous.

Remark 2.1. Consider the case where A generates an analytic semi-group $T(t)$ and $-A$ is of type $(\omega,\ M)$. Then the fractional power $(-A)^\alpha$, $\alpha \geq 0$, can be defined and

$$\limsup_{t \to 0+} t^\alpha \| T(t)(-A)^\alpha \| < \infty \quad \text{holds (see [12, p.69]).}$$

Let A_r, $r = 1,\cdots, m$, be linear combinations of fractional powers $\sum_{i=1}^{k_r} c_i^r (-A)^{\alpha_i^r}$, where c_i^r are constants and $0 \leq \alpha_i^r < 1$ ($i = 1,\cdots, k_r$). Then the assumptions H_1 and H_2^q with any q such that $q < \min \{ 1/\alpha_i^r :\ r = 1,\cdots, m,\ i = 1,\cdots, k_r \}$ are satisfied.

3. Construction and Representation of The Fundamental Solution.

In this section we construct the fundamental solution of the system S and give its explicit representation in terms of $T(t)$ and A_r.

Let $f = 0$, $g = 0$ and let the assumptions H_0, H_1 and H_2^1 (which is weaker than H_2^q for all $q \in (1, \infty]$) be satisfied. Then as in section 2, we can construct the mild solution $x(t;\, x_0) = x(t;\, x_0, 0, 0) \in X$ for any $x_0 \in X$. The mapping $G : R^+ \times X \to X$ defined by $G(t, x_0) = x(t;\, x_0)$ give rise to generates a one parameter family of bounded operators $\{ G(t) : t \geq 0 \}$, where $G(t)$ is defined by $G(t)x = G(t,x)$ for $x \in X$ and satisfies the following properties:

(i) $G(t) = T(t)$ for all $t \in [0, \tau]$ and $G(t) \in L(X)$ for all $t \geq 0$.

(ii) For each $x_0 \in X$, $G(t)x_0$ is continuous on R^+.

These are easy to verify.

Analogously to the finite dimensional case, we shall call $G(t)$ the fundamental solution of S. This terminology will be justified by Theorem 4.2 in the next section.

Let $x_0 \in X$ and

$$G_k(t) = G(t+(k-1)\tau) \qquad \text{for } t \in [0, \tau] \text{ and } k = 1, 2, \cdots . \tag{3.1}$$

Then by the definition of the mild solution, we obtain the relation for the operators $G_j(t)$, i.e. $G_j(t)x_0$, $j = 1, \cdots, k$ are given inductively by

$$G_j(t)x_0 = T(t)G_j(0)x_0 + \sum_{r=1}^{\min(j-1,m)} \int_0^t T(t-s)A_r G_{j-r}(s)x_0\, ds, \tag{3.2}$$

where $G_j(0)x_0 = G_{j-1}(\tau)x_0$. From (3.2) follows the next recursive formula for $G_k(t)$.

$$G_1(t) = T(t) \qquad \text{and}$$

$$G_k(t) = T(t)G_{k-1}(\tau) + \sum_{i=1}^{\min(k-1,m)} \int_0^t T(t-s)A_i G_{k-i}(s)\, ds \qquad \text{for } k \geq 2 \tag{3.3}$$

The expression of the formula (3.3) is formal. We now give the definite meaning of (3.3). We first consider $G_2(t)$. The operator $G_2(t)$ is given formally by

$$G_2(t) = T(t+\tau) + \int_0^t T(t-s)A_1T(s)ds.$$

BY assumptions H_0 and H_2^1, $T(t-s)A_1T(s)x$ is Bochner integrable on $[0, t]$ for any fixed $x \in X$ as in section 2. Hence the operator $F(t) = \int_0^t T(t-s)A_1T(s)ds$ can be defined by the equation $F(t)x = \int_0^t T(t-s)A_1T(s)xds$ for $x \in X$. To show that $F(t)$ is a bounded operator, we first prove that the mapping $\mathcal{F} : X \to L_1(0, t; X)$ defined by $(\mathcal{F}x)(s) = T(t-s)A_1T(s)x$, $s \in [0, t]$, is closed (see Dunford and Schwartz [5, p.685]). Let $x_n \to x$ in X and $\mathcal{F}x_n \to h$ in $L_1(0,t; X)$. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $(\mathcal{F}x_{n_j})(s) \to h(s)$ in X for a.a. $s \in [0, t]$. On the other hand $T(t-s)A_1T(s)x_n \to T(t-s)A_1T(s)x$ in X for a.a. $s \in [0, t]$ because of the boundedness of $T(t)$ and $T(t-s)A_1$. This shows that $(\mathcal{F}x)(s) = h(s)$ for a.a $s \in [0, t]$, and hence $\mathcal{F}$ is closed and therefore bounded. It is in this sense that the integral appearing in the formula of $G_2(t)$ should be interpreted. Furthermore $G_2(\cdot)x \in C(0,\tau; X)$ for each $x \in X$. We can prove by induction that the same is true for $G_k(t)$, $k \geq 3$.

Concerning with the formula (3.3), we define the operators $T_1(t),\cdots, T_k(t)$, $t \geq 0$, inductively by

$$T_1(t) = T(t) \qquad \text{and}$$

$$T_k(t) = \sum_{i=1}^{\min(k-1,m)} \int_0^t T(t-s)A_iT_{k-i}(s)ds \qquad \text{for } k = 2,3,\cdots . \tag{3.4}$$

Here the interpretation of the integrals in (3.4) is same as given in (3.3). Then by (3.2) and (3.4), $G_k(t)x_0$ can be written as

$$G_k(t)x_0 = T_k(t)x_1 + T_{k-1}(t)x_2 +\cdots + T_2(t)x_{k-1} + T_1(t)x_k,$$

where $x_j = G_j(0)x_0$, $j = 1,\cdots, k$.

The formula (3.4) is also inductive but simpler than (3.3), and from this formula we can derive the explicit form of $T_k(t)$. Before giving the form, let us define the index sets $\Lambda(j,k)$ for all $j = 1,2,\cdots$ and $k = 1,2,\cdots$ by

$$\Lambda(j,k) = \{ (i_1,\cdots, i_j) : 1 \le i_1,\cdots, i_j \le m \text{ and } i_1 + \cdots + i_j = k \}.$$

We then obtain from (3.4) the following integral expression of $T_k(t)$ for $k \geq 2$.

$$T_k(t) = \sum_{j=1}^{k-1} \sum_{\Lambda(j,k-1)} \int_0^t T(t-s_{j-1})A_{i_1} \cdots \int_0^{s_1} T(s_1-s)A_{i_j} T(s)ds ds_1\cdots ds_{j-1}. \quad (3.5)$$

The meaning of the iterated integrals appearing in (3.5) is similar to those given above. We note that $T_k(t)$ is strongly continuous on R^+ for each $k = 1,2,\cdots$.

Now, it is possible to give an expression of $G(t)$ in terms of $T_k(t)$. From (3.3), (3.4) and (3.5) it follows that

$$G(t) = \sum_{i=1}^{k} T_i(t-(i-1)\tau), \quad t \in [(k-1)\tau, k\tau], \quad (3.6)$$

which will be used in the next section.

Summing up the above arguments, we have the following theorem.

<u>THEOREM 3.1.</u> Let the assumptions H_0, H_1 and H_2^1 be satisfied. Then the set of one parameter family of strongly continuous operators

$$\{ T_k(\cdot) : k = 1,2,\cdots \}$$

can be constructed and is given by (3.5), and the fundamental solution $G(t)$ is given by (3.6).

<u>Example. 3.1.</u> Let X be a complex separable Hilbert space with inner product $\langle \ , \ \rangle$ and A is self-adjoint on X. Then the semi-group $T(t)$ generated by

A is also self-adjoint. Therefore, from Nagy's theorem it follows that there exists a resolution of unity $\{E(\cdot)\}$ on some half infinite interval $(-\infty, c]$ corresponding to $T(t)$ such that $T(t)$ is given by

$$T(t) = \int_{-\infty}^{c} e^{\lambda t} dE(\lambda) \tag{3.7}$$

provided that there exists $t > 0$ such that $T(t)x = 0$ implies $x = 0$ (cf. [6,7]). Now let $m = 1$ and $A_1 = I$, the identity operator on X. Then the fundamental solution $G(t)$ of (2.1) is given by

$$G(t) = \sum_{i=1}^{k} \frac{(t-(i-1)\tau)^{i-1}}{(i-1)!} \int_{-\infty}^{c} e^{\lambda(t-(i-1)\tau)} dE(\lambda), \qquad t \in [(k-1)\tau, k\tau].$$

Example 3.2. In addition to the assumptions in Example 3.1, we suppose that A has compact resolvent. Then there exists the set of eigenvalues and eigenfunctions $\{ \lambda_n, \psi_{nj} : j = 1,\cdots, m_n, n = 1,2,\cdots \}$ of A (Kato [8, p.277]). In this case the semi-group $T(t)$ in (3.7) is analytic and is represented by

$$T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} \sum_{j=1}^{m_n} \langle x, \psi_{nj} \rangle \psi_{nj}, \qquad t \geq 0 \qquad \text{for each } x \in X. \tag{3.8}$$

Let A and A_1 commute. Since A_1^i ($i = 1,2,\cdots$) commutes with $T(t)$ for $t \geq 0$, i.e., for any $x \in D(A_1^i)$ and $t \geq 0$, $T(t)x \in D(A_1^i)$ and $A_1^i T(t)x = T(t)A_1^i x$, the fundamental solution $G(t)$ is given by

$$G(t)x = \sum_{n=1}^{\infty} \sum_{i=1}^{k} \frac{(t-(i-1)\tau)^{i-1}}{(i-1)!} e^{\lambda_n (t-(i-1)\tau)} \sum_{j=1}^{m_n} \langle A_1^{i-1} x, \psi_{nj} \rangle \psi_{nj} ,$$

$$t \in [(k-1)\tau, k\tau] \tag{3.9}$$

for $x \in D(A_1^{k-1})$.

It is easy to verify the equality (3.9) by the expressions (3.5) and (3.6).

4. Representations of The Mild Solution.

In this section we shall give two different types of concrete representation of the mild solution without induction. The first one is expressed by the operators $T_k(t)$ and the second by the fundamental solution $G(t)$. The second one is well known as a variation of constants formula and is extended for more general delay systems both in finite dimensional space (Bellman and Cooke [1], Oğuztöreli [11]) and infinite dimensional space (Delfour and Mitter [3,4] in which all operators appearing in the system are bounded).

<u>THEOREM 4.1.</u> Let $x_0 \in X$, $f(\cdot) \in L_p^{loc}(R^+; X)$, $g(\cdot) \in L_{p'}(-m\tau,0; X)$ and let the assumptions H_0, H_1 and $H_2^{q'}$ with $1/p' + 1/q' = 1$ be satisfied. Then the mild solution $x(t; x_0,f,g)$ is given by

$$x(t; x_0,f,g) = \sum_{i=1}^{k} T_i(t-(i-1)\tau)x_0 + \sum_{i=1}^{k}\int_0^{t-(i-1)\tau} T_i(t-(i-1)\tau-s)f(s)ds$$
$$+ \sum_{i=1}^{k}\sum_{r=1}^{m}\int_{-r\tau}^{t-(i-1+r)\tau} (T_i(t-(i-1+r)\tau-s)A_r)\hat{g}(s)ds, \tag{4.1}$$

where $t \in [(k-1)\tau, k\tau]$ and $\hat{g}(s) = \begin{cases} g(s), & s \in [-m\tau, 0) \\ 0, & s \in [0, \infty) \end{cases}$.

This theorem can be proved by using mathematical induction. To give some definite meaning of the operator $(T_i(t)A_r)$ in (4.1), which is a bounded extension of $T_i(t)A_r$, Hausdorff-Young's inequality is used effectively. For detailed discussions, see Nakagiri [10].

We next give another representation of the mild solution in terms of $G(t)$ which is well known as a variation of constants formula and has a simpler form than that given in Theorem 4.1.

THEOREM 4.2. Under the same assumptions in Theorem 4.1, the mild solution is given by

$$x(t; x_0, f, g) = G(t)x_0 + \int_0^t G(t-s)f(s)ds + \sum_{r=1}^{m} \int_{-r\tau}^{0} (G(t-r\tau-s)A_r)g(s)ds, \qquad (4.2)$$

where $(G(s)A_r) = 0$ if $s < 0$.

Here the sense of $(G(s)A_r)$ is that given as $\sum_{i=1}^{j} (T_i(s-(i-1)\tau)A_r)$ if $s \in [(j-1)\tau, j\tau]$.

5. An Application.

The representation (4.2) is quite useful to obtain the fundamental theorems for the system S such as stability, continuous dependence and existence of periodic or almost periodic solutions as well as the system theoretical results such as controllability, observability, stabilizability, identifiability and existence of optimal controls [9,10]. Here we give an application of (4.2) to the concept of controllability.

Let U be a Banach space. In the system S, we put

$$f(t) = Bu(t), \quad u \in L_p^{loc}(R^+; U) \quad \text{and} \quad B \in L(U,X).$$

$L_p^{loc}(R^+; U)$ is the space of controls. First we shall give a definition of exact and approximate controllabilities of the system S. To define these concepts, the following set of attainability is needed.

$$A_t(x_0, L_p) = \{ x \in X : x = x(t; x_0, Bu, 0) \text{ where } u(\cdot) \in L_p^{loc}(R^+; U) \}.$$

Definition. The system S is said to be

(i) p-exactly controllable on $[0, t]$ if $A_t(x_0, L_p) = X$ for any $x_0 \in X$;

(ii) p-approximately controllable on $[0, t]$ if $\overline{A_t(x_0, L_p)} = X$ for any $x_0 \in X$

The following theorems can be established via infinite dimensional linear system theory (cf. Curtain and Pritchard [2]).

THEOREM 5.1. Let X and U be reflexive Banach spaces and let $p \in (1,\infty)$. Then the system S is p-exactly controllable on $[0, t]$ if and only if there exists $K_t > 0$ such that

$$\| x^* \|_{X^*} \le K_t \| B^*G(\cdot)^*x^* \|_{L_q(0,t;\, U^*)}, \qquad \forall x^* \in X^*,$$

where $1/p + 1/q = 1$.

THEOREM 5.2. Let X be infinite dimensional and let $p \in [1, \infty]$. Then the system S is never p-exactly controllable on any $[0, t]$, $t > 0$ if one of the following conditions holds:

(i) $T(t)$ is compact for all $t > 0$.

(ii) X has a Schauder basis and the operator B is compact.

THEOREM 5.3. The system S is p-approximately controllable on $[0, t]$ if and only if $B^*G(s)^*x^* = 0$ in U^* for all $s \in [0, t]$ implies $x^* = 0$ in X^*.

Now we consider the same system given in Example 3.2.

$$S_c : \quad \begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-\tau) + \sum_{\nu=1}^{K} b_\nu u_\nu(t), & t > 0 \\ x(0) = x_0, \quad x(s) = 0, & s \in [-\tau, 0), \end{cases}$$

where $b_\nu \in D(A_1^\infty)$, $u_\nu(\cdot) \in L_p^{loc}(R^+)$ ($\nu = 1,\cdots, K$) ($D(A_1^\infty) = \bigcap_{i=1}^{\infty} D(A_1^i)$).

In this case $U = C^K = \{ (u_1,\cdots, u_K) \}$ and $B \in L(C^K, X)$ is given by $B(u_1,\cdots, u_K) = \Sigma_{\nu=1}^{K} b_\nu u_\nu$. For each natural numbers n and i and each K-tuple b_ν, we define the $m_n \times K$ matrix B_n^i by

$$B_n^i = \begin{pmatrix} \langle A_1^{i-1} b_1, \psi_{n1} \rangle & \cdots & \langle A_1^{i-1} b_K, \psi_{n1} \rangle \\ \langle A_1^{i-1} b_1, \psi_{n2} \rangle & \cdots & \langle A_1^{i-1} b_K, \psi_{n2} \rangle \\ \vdots & & \vdots \\ \langle A_1^{i-1} b_1, \psi_{nm_n} \rangle & & \langle A_1^{i-1} b_K, \psi_{nm_n} \rangle \end{pmatrix}$$

Then from Theorem 5.3 and the representation (3.9), we have the following result.

COROLLARY 5.4. Let $b_\nu \in D(A_1^\infty)$ and $u_\nu \in L_p^{loc}(R^+)$ ($\nu = 1, \cdots, K$) in S_c. Then the system S_c is p-approximately controllable on $[0, t]$, $t \in [(k-1)\tau, k\tau)$ if and only if $\text{rank}\ [B_n^1, \cdots, B_n^k] = m_n$ for all $n = 1, 2, \cdots$.

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