

Periodic solutions of functional differential equations

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1. Introduction. The purpose of this paper is to discuss the existence problem of periodic solutions of functional differential equations. Particularly, we are concerned with a reduction of a given ω -periodic functional differential equation with the delay $r > \omega$ to an auxiliary ω -periodic functional differential equation with the delay ω . Moreover, we obtain a Razumikhin type theorem concerning the existence of ω -periodic solutions of functional differential equations by using a strongly convex Liapunov function.

Let R^+ and R denote the interval $0 \leq t < \infty$ and $-\infty < t < \infty$, respectively. For a given r , $0 < r \leq \infty$, C_r denotes the Banach space of continuous and bounded functions defined by

$$C_r = \{ \phi : [-r, 0] \rightarrow R^n, \text{ continuous} \}, \quad 0 < r < \infty,$$

or

$$C_\infty = \{ \phi : (-\infty, 0] \rightarrow R^n, \text{ continuous and bounded} \},$$

with the uniform norm, $|\phi| = \sup \{ |\phi(\theta)| : -r < \theta \leq 0 \}$, where $|\cdot|$ denotes the usual Euclidean norm in R^n . For a given continuous function $x(s)$, the symbol x_t will denote the element of C_r such that $x_t(\theta) = x(t+\theta)$, $-r < \theta \leq 0$.

Let $f(t, \phi) : R \times C_r \rightarrow R^n$ be a completely continuous function

which is ω -periodic in t , that is, $f(t+\omega, \phi) = f(t, \phi)$ for all $(t, \phi) \in \mathbb{R} \times C_r$ and some positive constant ω .

Consider a functional differential equation

$$(1) \quad \dot{x}(t) = f(t, x_t),$$

where $\dot{}$ denotes the right hand derivative.

2. Reductions of Equation (1). In this section, we shall consider reductions of Equation (1) with $r > \omega$ to auxiliary ω -periodic functional differential equations with the delay ω . Let $\sigma_t(\psi) : \mathbb{R} \times C_\omega \rightarrow C_r$ be a mapping such that $\sigma_t(\psi)$ is ω -periodic in t , continuous on $\mathbb{R} \times C_\omega$, takes bounded sets in $\mathbb{R} \times C_\omega$ into bounded sets, $\sigma_t(\psi)(\theta)$ is ω -periodic in θ on $(-r, 0]$ if $\psi(0) = \psi(-\omega)$, and that $\sigma_t(\psi)(\theta)$ is ω -periodic in θ on $(-r, -\omega]$ if $r > 2\omega$. A simple example is :

$$(2) \quad \sigma(\psi)(\theta) = \begin{cases} \psi(\theta) \\ \psi(\theta+k\omega) - \frac{\theta+(k+1)\omega}{\omega}(\psi(0) - \psi(-\omega)), \quad -\min\{r, (k+1)\omega\} \leq \theta < -k\omega, k \geq 1. \end{cases}$$

For such a mapping $\sigma_t(\psi)$, let $g(t, \psi) : \mathbb{R} \times C_\omega \rightarrow \mathbb{R}^n$ be a function defined by

$$(3) \quad g(t, \psi) = f(t, \sigma_t(\psi)).$$

Clearly $g(t, \psi)$ is completely continuous, and ω -periodic in t . For $g(t, \psi)$ defined by (3), consider an auxiliary equation

$$(4) \quad \dot{x}(t) = g(t, x_t).$$

Then this equation always has a solution for the initial value problem, while Equation (1) may fail to have a solution for some initial value problem (see Seifert [3]). And we have the following theorem.

Theorem 1. An ω -periodic solution $x(t)$ ($t \in \mathbb{R}$) of Equation (1) is a solution of Equation (4), and vice versa.

The proof of this theorem is clear from the definition of $g(t, \psi)$ and the properties of $\sigma_t(\psi)$.

3. Existence of periodic solutions of Equation (1). In this section, we shall show the existence of ω -periodic solutions of (1) via the existence of ω -periodic solutions of (4). Here we consider a strongly convex Liapunov function defined by strengthening the conditions for a convex Liapunov function in [2].

A function $V(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a Liapunov function if $V(t, x)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ and satisfies $V(t, x) \geq a(|x|)$ for a continuous function $a(u)$ such that $a(u) \rightarrow \infty$ as $u \rightarrow \infty$. We shall say a Liapunov function $V(t, x)$ is convex if for each fixed $t \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : V(t, x) \leq k\}$ is convex in \mathbb{R}^n . Moreover, a convex Liapunov function $V(t, x)$ is said to be strongly convex if for each fixed $t \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : V(t, x) = k\}$ is the boundary of $\{x \in \mathbb{R}^n : V(t, x) \leq k\}$. We define

$V'_{(1)}(t, \phi)$ by

$$V'_{(1)}(t, \phi) = \limsup_{\tau \rightarrow 0^+} \frac{1}{\tau} \{V(t+\tau, x(t+\tau, t, \phi)) - V(t, \phi(0))\}.$$

Clearly we have

$$(5) \quad V'_{(1)}(t, \phi) = \limsup_{\tau \rightarrow 0^+} \frac{1}{\tau} \{V(t+\tau, \phi(0)+\tau f(t, \phi)) - V(t, \phi(0))\}$$

if $V(t, x)$ is locally Lipschitzian with respect to x .

Now consider the following scalar equation

$$(6) \quad \dot{u} = h(t, u),$$

where $h(t, u) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and locally Lipschitzian with respect to u .

The following theorem is given in [1], where the delay r is finite.

Theorem 2. Let $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a continuous, ω -periodic, convex Liapunov function, and let $L : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, non-decreasing function such that

$$(7) \quad L(u) > u \text{ for all } u > 0,$$

or

$$(8) \quad L(u) = u \text{ for all } u > 0.$$

Suppose that there exists a continuous function $u(t)$ defined on $[t_0-r, t_0+\omega]$ for some t_0 such that $u(t)$ is a solution of (6) on $[t_0, t_0+\omega]$ which satisfies $u(t_0+\omega+\theta) \leq u(t_0+\theta)$ for $-r \leq \theta \leq 0$, $u(t) \geq V(t, 0)$ for $t_0 \leq t \leq t_0+\omega$, and $u(t+\theta) \leq L(u(t))$ for $t_0 \leq t \leq t_0+\omega$, $-r \leq \theta \leq 0$, and that we have

$$(9) \quad V'_{(1)}(t, \phi) \leq h(t, V(t, \phi(0)))$$

for all functions $\phi \in C_r$ with the property that

(10) $V(t, \phi(0)) \geq u(t)$, $V(t+\theta, \phi(\theta)) \leq L(V(t, \phi(0)))$ for $-r \leq \theta \leq 0$.

Then Equation (1) has an ω -periodic solution.

Now we can obtain the following theorem concerning the existence of an ω -periodic solution of Equation (1) with $r = \infty$ by combining Theorem 1 and Theorem 2.

Theorem 3. Let $V: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a continuous, ω -periodic in t , strongly convex Liapunov function which is locally Lipschitzian with respect to x , and let $L: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing continuous function which satisfies (7) or (8). Suppose that Equation (6) has a solution $u(t)$ on $[0, \omega]$ such that $u(0) \geq u(\omega)$, $u(t) > V(t, 0)$ for $0 \leq t \leq \omega$, $\bar{u} \leq L(\underline{u})$ where $\bar{u} = \max_{0 \leq t \leq \omega} u(t)$, $\underline{u} =$

$\min_{0 \leq t \leq \omega} u(t)$, and that we have

$$(11) \quad V'_{(1)}(t, \phi) \leq h(t, V(t, \phi(0)))$$

for all functions $\phi \in C_\infty$ with the property that

$$(12) \quad \phi \in P_\infty, \quad V(t, \phi(0)) \geq u(t), \quad V(t+\theta, \phi(\theta)) \leq L(V(t, \phi(0))) \quad \text{for } -\infty < \theta \leq 0,$$

where $P_\infty = \{ \phi \in C_\infty : \phi(\theta) \text{ is } \omega\text{-periodic on } (-\infty, -\omega] \}$. Then Equation (1) has an ω -periodic solution $x(t)$ such that $V(t, x(t)) \leq u(t)$ for $0 \leq t \leq \omega$.

This theorem can be proved in the following manner. First, we define a function $\bar{\sigma}_t(\psi) : \mathbb{R} \times C_\omega \rightarrow C_\infty$ which is similar to $\sigma_t(\psi)$ in Section 2 by using the strongly convex Liapunov function $V(t, x)$. For any $(t, \psi) \in \mathbb{R} \times C_\omega$, let $v = \max_{-\omega \leq \theta \leq 0} V(t+\theta, \psi(\theta))$ and

$v^* = \max \{ v, L(\underline{u}) \}$. Define a set Σ^* by

$$\Sigma^* = \{ (s, x) \in \mathbb{R} \times \mathbb{R}^n : V(s, x) \leq v^* \}.$$

Now let $\sigma(\psi)$ be a function defined by (2). For $\theta \leq 0$ such that $(t+\theta, \sigma(\psi)(\theta)) \in \Sigma^*$, define $\bar{\sigma}_t(\psi)(\theta)$ by $\bar{\sigma}_t(\psi)(\theta) = \sigma(\psi)(\theta)$. For $\theta < -\omega$ such that $(t+\theta, \sigma(\psi)(\theta)) \notin \Sigma^*$, define $\bar{\sigma}_t(\psi)(\theta)$ by $\bar{\sigma}_t(\psi)(\theta) = \lambda(\theta)\sigma(\psi)(\theta)$, where $\lambda(\theta)$ is a uniquely determined number such that $\lambda(\theta) \in (0, 1)$ and $V(t+\theta, \lambda(\theta)\sigma(\psi)(\theta)) = v^*$. Clearly the function $\bar{\sigma}_t(\psi)$ has the same properties with $\sigma_t(\psi)$ in Section 2. Moreover, we have $V(t+\theta, \bar{\sigma}_t(\psi)(\theta)) \leq v^*$ for $\theta \leq 0$ by the definition of $\bar{\sigma}_t(\psi)$. Using this $\bar{\sigma}_t(\psi)$, define an auxiliary function $g^*(t, \psi)$ by

$$(13) \quad g^*(t, \psi) = f(t, \bar{\sigma}_t(\psi)),$$

and consider the following auxiliary equation

$$(14) \quad \dot{x}(t) = g^*(t, x_t).$$

If we show that

$$(15) \quad V_{(14)}'(t, \psi) \leq h(t, V(t, \psi(0)))$$

under the condition

$$(16) \quad V(t, \psi(0)) \geq u(t), \quad V(t+\theta, \psi(\theta)) \leq L(V(t, \psi(0))) \quad \text{for } -\omega \leq \theta \leq 0,$$

then we can apply Theorem 2 to Equation (14). Let (t, ψ) satisfy (16), and let $v = \max_{-\omega \leq \theta \leq 0} V(t+\theta, \psi(\theta))$ and $v^* = \max \{ v, L(\underline{u}) \}$. If we take $\phi = \bar{\sigma}_t(\psi)$ for (t, ψ) , then we obtain

$$V(t+\theta, \phi(\theta)) \leq v^* \leq L(V(t, \phi(0))) \text{ for } -\infty < \theta \leq 0.$$

Thus (16) implies (12), and consequently we have (11), which is not different from (15) by (5) and (13), since $V(t, x)$ is locally Lipschitzian with respect to x .

Next, $u(t)$ can be extended continuously on $[-\omega, \omega]$ by defining $u(t) = \max\{u(0), u(t+\omega)\}$ for $-\omega \leq t \leq 0$. Then all assumptions of Theorem 2 are satisfied with this $u(t)$ and $t_0 = 0$, and Equation (14) has an ω -periodic solution. Thus we can obtain the conclusion of this theorem by Theorem 1.

Finally, we present an application of Theorem 3. Consider a scalar ω -periodic functional differential equation

$$(17) \quad \dot{x}(t) = -x(t) + 4x(t-\omega) - 4x(t-2\omega) + f(t, x_t),$$

where $f(t, \phi) : \mathbb{R} \times C_\infty \rightarrow \mathbb{R}$ is completely continuous, ω -periodic in t , and satisfies

$$(18) \quad |f(t, \phi)| \leq |\phi| \text{ if } t \in \mathbb{R}, |\phi(0)| > K$$

for some positive constant K . If we take $L(u) = u$ and $V(x) = \frac{x^2}{2}$, then V is a strongly convex Liapunov function which is locally Lipschitzian, and we have

$$V'_{(17)}(t, \phi) = -\phi^2(0) + \phi(0)f(t, \phi) \leq -\phi^2(0) + \phi^2(0) \frac{|\phi|}{|\phi(0)|} \frac{|f(t, \phi)|}{|\phi|} \leq 0$$

under the condition (12) if the solution $u(t)$ of (6) with $h(t, u) \equiv 0$ satisfies $u(t) > \frac{K^2}{2}$, where K is the one in (18). Therefore, by Theorem 3, there exists an ω -periodic solution of (17).

Suppose that (17) is an equation with the finite delay r . In this case, we cannot apply Theorem 37.1 in [4] to (17) to

conclude that (17) has an ω -periodic solution, since r is greater than the period ω . Moreover, $V'_{(17)}(t, \phi)$ cannot be compared with $h(t, V(t, \phi(0))) \equiv 0$ under the condition (10). Thus we have no idea how to apply Theorem 2.

References

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