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On skew group rings

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Let R be a ring with 1, and G a group. $U(R)$ denotes the group of units of R. Given maps $\alpha: G \to \mathrm{Aut}(R)$ and $\gamma: G \times G \to U(R)$ such that

(i) $\gamma(g,h)\gamma(gh,i) = \gamma(h,i)^{\alpha(g)^{-1}}\gamma(g,hi)$

and

(ii) $\gamma(g,h)r^{\alpha(gh)^{-1}} = r^{\alpha(g)^{-1}\alpha(g)^{-1}}\gamma(g,h)$

for all $g, h, i \in G$, $r \in R$, we define the crossed product $R*G$ to be a free R-module with basis $\{\bar{g} \mid g \in G\}$. The multiplication is given by the rule

$$(r_g\bar{g})(r_h\bar{h}) = r_g r_h^{\alpha(g)^{-1}}\gamma(g,h)\overline{gh}.$$

This makes $R*G$ an associative ring with unit element $\gamma(1,1)^{-1}\bar{1}$. The map $r \to r\gamma(1,1)^{-1}\bar{1}$ is a ring monomorphism of R into $R*G$. We therefore consider R as a subring of $R*G$. If $\gamma(g,h) = 1$ for all $g, h \in G$, $R*G$ called a skew group ring, and if $\alpha(g) = 1$ for all $g \in G$, $R*G$ is called a twisted group ring.

Let S be any ring. Let $R*G$ be a crossed product with G a finite group, and V, V' $(S,R*G)$-modules. For $g \in G$ and $k \in \mathrm{Hom}_{(S,R)}(V,V')$, we define $k^g(v) = k(v\bar{g}^{-1})\bar{g}$ for all $v \in V$. One may check that $k \to k^g$ defines a group action of G on $\mathrm{Hom}_{(S,R)}(V,V')$. It is clear that the fixed submodule is $\mathrm{Hom}_{(S,R*G)}(V,V')$. Therefore $t_G(k) = \sum_{g\in G} k^g$ is an $(S,R*G)$

omomorphism for every $k \in Hom_{(S,R)}(V,V')$. If there exists an $h \in End_{(S,R)}(V')$ such that $t_G(h) = 1_{V'}$, then $\hat{k} = t_G(hk)$ is an $(S,R*G)$-homomorphism and $\hat{k} = k$ on every $R*G$-submodule of V. $C(R)$ denotes the center of R. If there exists an element $c \in C(R)$ such that $t_G(c) = \sum_{g \in G} c^{\alpha(g)} = 1$, then $t_G(T_c) = 1$, where $T_c \in End_{(S,R)}(V')$ denotes right multiplication by c. If V' is $|G|$-torsion free and $V \cdot |G|$, then we can define an element $h \in End_{(S,R)}(V')$ by $h(v) = |G|^{-1}v$ for all $v \in V'$. Clearly, $t_G(h) = 1$.

Now, the proof of the following is easy.

Proposition 1. Let $W \subset V$ be $(S,R*G)$-modules. Suppose there exists an element $c \in C(R)$ such that $t_G(c) = 1$. If $W \mathrel{<\!\!\oplus} {}_SV_R$, then $W \mathrel{<\!\!\oplus} {}_SV_{R*G}$.

We note that if the order of G is invertible in R, then $|G|^{-1} \in C(R)$ and $t_G(|G|^{-1}) = 1$.

A ring R is said to be fully right idempotent if every right ideal of R is idempotent. For example, von Neumann regular rings, right V-rings, and ring which are direct sum of simple rings, are fully right idempotent.

Corollary 1. Let $R*G$ be a crossed product with G a finite group. Suppose there exists an element $c \in C(R)$ such that $t_G(c) = 1$.

(1) If R is fully right idempotent, then so is $R*G$.

(2) If R is regular, then so is $R*G$.

(3) If R is a right V-ring, then so is $R*G$.

(4) If R is a direct sum of simple rings, then so is $R*G$.

Proof. We prove only (3). If K is a maximal right ideal of $R*G$, then there exists a maximal submodule M of $R*G_R$ which contains K. It is easy to see that $\bigcap_{g\in G} M\bar{g} = K$. Therefore $P = R*G/K$ is a direct sum of simple right R-modules, and hence P is an injective R-module. Let E be an injective hull of P_{R*G}. Since $P \oplus\!\!<\, E_R$, $P = E$ by Proposition 1.

Let $G \subset \mathrm{Aut}(R)$ be a finite group, and $R*G$ a skew group ring. R can be viewed as a right $R*G$-module by denining $r\cdot\sum x_g g = \sum(rx_g)^g$; for x_g and r in R. If we set $f = \sum_{g\in G} g$, then $R \simeq fR*G$ as right $R*G$-modules. The fixed subring is denoted by R^G; $R^G = \{r \in R \mid r^g = r$ for all $g \in G\}$. For a module V over a ring S, let $L(V_S)$ denote the lattice of S-submodules of V.

Lemma 1. Let $G \subset \mathrm{Aut}(R)$ be finite. Suppose there is an element $c \in R$ such that $t_G(c) = 1$. Then the following are equivalent:

1) R_{RfR} is s-unital; that is, $r \in r\cdot RfR$ for all $r \in R$.

2) $L(R^G{}_{(R^G)}) \to L(R_{R*G})$; $I \to IR$, is a lattice isomorphism.

Proof. Since $t_G(r\cdot R*G)R = r\cdot RfR$ for every $r \in R$, the assertion is clear.

Corollary 2. Let R be a fully right idempotent ring, and $G \subset \mathrm{Aut}(R)$ finite. Suppose there is an element $c \in C(R)$ such that $t_G(c) = 1$. Then the lattice of right ideals of R^G is isomorphic to the lattice of G-invariant right ideals of R.

Proof. We set $Q = R*G$. By the part (1) of Corollary 1, Q is fully right idempotent. Let r be an element of R. Then, $fr \in (frQ)^2 \subset fr(RfR)$ and so $r \in r\cdot(RfR)$.

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