数理解析研究所講究録
第 438 巻 1981 年 66-70

ON GROUP ALGEBRAS OF FINITE GROUPS

Shigeo KOSHITANI

Department of Mathematics, Faculty of Science
Chiba University, Chiba-city, 260, Japan

In this note we study the group algebra KG of a finite p-solvable group G over a field K of characteristic $p > 0$. Let $J(KG)$ be the Jacobson radical of KG, and let $t(G)$ be the least positive integer t such that $J(KG)^t = 0$. Since $J(EG) = E \otimes_K J(KG)$ for any extension field E of K, we may assume that K is algebraically closed. We would like to know the relation between $t(G)$ and the structure of G. When $t(G) \leqslant 3$, p-solvable groups G are completely determined by D.A.R. Wallace ([9], [10]) and K. Motose and Y. Ninomiya [7]. The purpose of this note is to determine the structure of p-solvable groups G with $t(G) = 4$ under the assumption that $O_{p'}(G)$ are abelian.

We shall use the following notation. For a positive integer n let S_n and A_n be the symmetric group and the alternating group of degree n, respectively. Let $O_{p'}(G)$ and $O^{p'}(G)$ be the maximal normal subgroup of G of order prime to p and the minimal normal subgroup of G of index prime to p, respectively. Following custom we write $O(G)$ and $O'(G)$ for $O_{2'}(G)$ and $O^{2'}(G)$, respectively. For a ring R and a positive integer n let $(R)_n$ be the ring of all $n \times n$ matrices with entries in R. We use the other notation following Gorenstein's book [3].

By making use of [9, Theorem], [2, Theorem 1] and [10, Theorem 3.3] we have

Proposition 1. If G is a finite p-solvable group with a p-Sylow subgroup P and if $t(G) = 4$, then $p = 2$ and one of the following holds;

(i) P is cyclic of order 4,

(ii) P is elementary abelian of order 8,

(iii) $G/O(G) \cong S_4$.

Remark 1. The converse of Proposition 1 does not hold in general (see Motose's example [6, Example 2]). However, the following holds.

Proposition 2. If $p = 2$ and if G is a finite 2-solvable group with a 2-Sylow subgroup P which satisfies one of the following;

(i) P is cyclic of order 4,

(ii) P is elementary abelian of order 8,

(iii) $G = S_4$,

then $t(G) = 4$.

Because of Propositions 1 and 2 we assume in the rest of this note that

$$p = 2 \quad \text{and} \quad G/O(G) \cong S_4.$$

Then a 2-Sylow subgroup P of G is dihedral of order 8. Thus, by [3, Theorem 7.7.3], P has subgroups X and Y such that X and Y are both noncyclic of order 4, $X \underset{P}{\not\sim} Y$, $|N_G(X):C_G(X)| = 6$ and $|N_G(Y):C_G(Y)| = 2$.

By [4, V 25.12 Satz, V 25.7 Satz und V 25.3 Satz], [8, Lemma 2.1] and [11, Proposition 3.2], we have

Lemma 1. If U is a subgroup of S_4 and if K^cU is a twisted group algebra of U over K with respect to the factor set c, then $K^cU \cong KU$ as K-algebras.

By making use of Lemma 1, [5, Theorem 2] and [1] we obtain the following two lemmas.

Lemma 2. $t(G) = 4$ if and only if $t(N_G(X)) = 4$.

Lemma 3. If $X \triangleleft G$, then

$$KG \cong \left(\bigoplus_{i=1}^{m} (KS_4)_{\alpha_i}\right) \oplus \left(\bigoplus_{j=1}^{n/2} (KA_4)_{\beta_j}\right) \oplus \left(\bigoplus_{k=1}^{u/3} (KP)_{\gamma_k}\right) \oplus \left(\bigoplus_{\ell=1}^{v/6} (KX)_{\delta_\ell}\right)$$

as K-algebras for positive integers α_i, β_j, γ_k and δ_ℓ where m, n, u and v are the numbers of irreducible complex characters ψ of $O(G)$ such that $I_G(\psi)/O(G) \cong S_4$, A_4, P and X, respectively, and $I_G(\psi)$ is the inertia group of ψ in G.

From the above lemmas we have the following main result.

Theorem. Let $M = O'(N_G(X))$. If $O(M)$ is abelian, then the following are equivalent:

(1) $t(G) = 4$.

(2) $t(M) = 4$.

(3) $|C_M(P)| = 2$ where P is a 2-Sylow subgroup of M.

(4) When $g \in M$ such that $|gO(M)| = 3$ in $M/O(M)$, we have $g \in C_M(O(M))$.

Remark 2. In Theorem for the case where $O(M)$ is nonabelian (2) and (3) are not equivalent in general.

REFERENCES

[1] R. Brauer, On 2-blocks with dihedral defect groups, Symposia Mathematica 13 (1974), 367-393.

[2] E.C. Dade, Blocks with cyclic defect groups, Ann. of Math. (2) 84 (1966), 20-48.

[3] D. Gorenstein, Finite Groups (Harper and Row, New York, 1968).

[4] B. Huppert, Endliche Gruppen I (Springer, Berlin, 1967).

[5] K. Morita, On group rings over a modular field which

possess radicals expressible as principal ideals, Science Reports of Tokyo Bunrika Daigaku A4 (1951), 177-194.

[6] K. Motose, On the nilpotency index of the radical of a group algebra II, Math. J. Okayama Univ. 22 (1980), 141-143.

[7] K. Motose and Y. Ninomiya, On the nilpotency index of the radical of a group algebra, Hokkaido Math. J. 4 (1975), 261-264.

[8] D.S. Passman, Radicals of twisted group rings, Proc. London Math. Soc. (3) 20 (1970), 409-437.

[9] D.A.R. Wallace, Group algebras with radicals of square zero, Proc. Glasgow Math. Assoc. 5 (1962), 158-159.

[10] D.A.R. Wallace, Lower bounds for the radical of the group algebra of a finite p-soluble group, Proc. Edinburgh Math. Soc. 16 (1968/69), 127-134.

[11] K. Yamazaki, On projective representations and ring extensions of finite groups, J. Faculty of Science of University of Tokyo 10 (1964), 147-195.