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Similarity and Universality of Turbulence

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Turbulent motion of an incompressible viscous fluid is mathematically represented by apparently irregular solutions of the Navier-Stokes equation. The essential feature of this motion lies in the singular role of the viscosity ν. In the absence of the viscosity $\nu = 0$, the fluid is a conservative system and a thermal equilibrium is possible for turbulence. In the presence of a small but nonzero viscosity $\nu > 0$, however, the fluid is dissipative and turbulence cannot be in thermal equilibrium. Then, what we can expect for turbulence is at most an "equilibrium" of different kind.

Kolmogorov (1941) proposed an equilibrium state that is determined by two parameters: the viscosity ν and the rate of energy dissipation $\varepsilon = -\dot{\mathcal{E}}$ where $\mathcal{E} = \langle|\vec{u}|^2\rangle/2$ represents the energy of turbulence. It immediately follows from this premise and dimensional analysis that the energy spectrum $E(k)$, k being the wavenumber, takes the following similarity form at wavenumbers much larger than the energy-containing wavenumber k_0:

$$E(k) = \varepsilon^{1/4}\nu^{5/4}F(k/k_d), \quad k_d = \varepsilon^{1/4}\nu^{3/4}, \tag{1}$$

where F is a nondimensional function and k_d represents the energy-dissipation wavenumber. In the inviscid limit $\nu \to 0$, the similarity form (1) leads to the well-known inertial subrange spectrum,

$$E(k) = C\varepsilon^{2/3}\, k^{-5/3}, \quad C = \text{const.}, \tag{2}$$

which characterizes Kolmogorov's <u>universal equilibrium</u> state.

In an incompressible viscous fluid, the energy dissipation ε is expressed as

$$\varepsilon = -\dot{\mathcal{E}} = 2\nu\mathcal{Q}, \tag{3}$$

where $\mathcal{Q} = \langle|\mathrm{rot}\,\vec{u}|^2\rangle$ is called the enstrophy. Then, Kolmogorov's premise,

$$\varepsilon > 0 \quad \text{as } \nu \to 0, \tag{4}$$

requires the divergence of $\mathcal{Q}$ or the enstrophy catastrophe in this limit:

$$\mathcal{Q} \to \infty \quad \text{as } \nu \to 0. \tag{5}$$

Thus, the validity of Kolmogorov's spectra (1) and (2) is entirely dependent on the existence of the enstrophy catastrophe.

It is obvious that the enstrophy catastrophe (5) does not occur in two-dimensional turbulence, in which the enstrophy dissipation η is expressed as

$$\eta = -\dot{\mathcal{Q}} = 2\nu\mathcal{P}, \tag{6}$$

where $\mathcal{P} = \langle|\mathrm{rot\,rot}\,\vec{u}|^2\rangle$ is called the palinstrophy. It follows from (6) that $\mathcal{Q}$ is finite if so initially so that

$$\varepsilon \to 0 \quad \text{as } \nu \to 0. \tag{7}$$

Obviously, Kolmogorov's theory is not applicable to this turbulence.

Batchelor (1969), Kraichnan (1967) and Leith (1968) proposed a different equilibrium state for two-dimensional turbulence which is determined by η and ν instead of ε and ν for three-dimensional turbulence. Under this premise the dimensional analysis yields the following similarity form:

$$E(k) = \eta^{1/6}\nu^{3/2}F(k/k_d), \quad k_d = \eta^{1/6}\nu^{-1/2}, \tag{8}$$

for $k \simeq k_d$, where k_d now represents the enstrophy dissipation wavenumber. In the inviscid limit $\nu \to 0$, the similarity form (8) leads to the two-dimensional inertial subrange spectrum,

$$E(k) = C'\eta^{2/3}k^{-3}, \quad C' = \text{const.}, \tag{9}$$

for $k_0 \ll k \ll k_d$.

In order that the above proposals of the equilibrium state are accepted as realistic, their compatibility with the Navier-Stokes equation must be examined. The statistical description of turbulence is most completely made in terms of Hopf's equation for the characteristic functional. It can be shown that the modified zero-fourth cumulant approximation yields exact similarity laws of the energy spectrum which are free from the assumption employed.

For three-dimensional turbulence, this approximation yields an <u>inviscid similarity</u> of the energy spectrum in the energy-containing range $k \simeq k_0$ and <u>Kolmogorov's similarity</u> in the energy-dissipation range $k \simeq k_d$, both of which are exact results. Thus, Kolmogorov's similarity law (1) together with its premise (4) is found to be an exact consequence of the mathematical equation governing turbulence. Numerical calculation of the evolution of the energy $\mathcal{E}$ in time starting from finite initial values of $\mathcal{E}$ and $\mathcal{Q}$ shows that the <u>enstrophy catastrophe</u> occurs at a finite critical time $t = t_c$, and the energy dissipation ε is zero and nonzero in the inviscid limit for the periods $t < t_c$ and $t > t_c$ respectively.

For two-dimensional turbulence, the same approximation yields an <u>inviscid similarity</u> in the energy-containing range $k \simeq k_0$ and <u>Batchelor's similarity</u> (8) in the enstrophy-dissipation range $k \simeq k_d$, both of which are also exact results. The energy $\mathcal{E}$ is conserved in the inviscid limit, but the enstrophy $\mathcal{Q}$ is kept constant in the same limit only before a critical time $t < t_c$ and decays thenceforth $t > t_c$. The last result confirms the validity of the premise adopted by Batchelor and others. Unlike the three-dimensional case, the critical time t_c for two-dimensional turbulence is shown to increase with decreasing viscosity ν as

$$t_c \propto [\log(1/\nu)]^{1/2}. \quad (10)$$

Hence, the <u>palinstrophy catastrophe</u>

$$\mathcal{P} \to \infty \quad \text{as} \quad \nu \to 0 \quad (11)$$

takes place for two-dimensional turbulence only after infinite time $t > t_c \to \infty$.