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On saddle-connection curves of analytic dynamical systems

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Let $f : R^2 \to R^2$ be an analytic diffeomorphism of the plane. We call a point x in R^2 a saddle point of f if x is a fixed point of f and that one of the absolute values of the eigenvalues of the Jacobian matrix df at the point is greater than one and the other is smaller than one.

For a saddle point x in R^2, we denote by W_x^s (resp. by W_x^u) the stable manifold (resp. the unstable manifold) associated to x.

Let E_x^s (resp. E_x^u) denote the eigenspace of linear map $df_x : T_xR^2 \to T_xR^2$ corresponding to the eigenvalue whose absolute value is smaller than one (resp. greater than one). Stable manifold W_x^s (resp. unstable manifold W_x^u) is an injectively immersed one dimensional manifold and is tangent to E_x^s (resp. E_x^u) at the saddle point.

Let $h : I \to R^2$ be an embedding of the unit interval into R^2. We call h a saddle connection if the following conditions i), ii) and iii) are satisfied.

i) the image of two boundary points $p = h(0)$ and $q = h(1)$ are saddle points of f.

ii) the image $h(I-\{0,1\})$ contains no fixed point of f.

iii) the image $h(I)$ is invariant under f.

We have the following theorem.

Theorem 1 If an analytic diffeomorphism $f : R^2 \to R^2$ can be extended to an automorphism $F : C^2 \to C^2$ of the two dimensional complex vector space considered as complex manifold, then there does not exist any saddle connection.

Proof We shall prove the theorem by contradiction. Suppose there was a saddle connection $h : I \to R^2$. In the first place, we assume that saddle points $p = h(0)$ and $q = h(1)$ are distinct. As h is an embedding and the image $h(I)$ is invariant under f, there is a neighborhood of 0 in I such that its image by h is included in W_p^s or W_p^u. We can assume, without loss of generality, it is included in W_p^u, the unstable manifold of p. In the contrary case, we can replace f by its inverse diffeomorphism f^{-1}. Embedding h is a diffeomorphism of the unit interval onto its image $h(I)$. Let $g : I \to I$ be the continuous mapping of the unit interval into itself defined by $g = h^{-1} \circ f \circ h$. Since h and f are diffeomorphisms, g is also a diffeomorphism. The image $h(I-\{0,1\})$ contains no fixed point of f by the assumption. Hence g has no fixed point in the interior of I. The boundary points 0 and 1 of I are fixed points of g. We see that

$$\frac{dg}{dt}(0) > 1$$

since some neighborhood of 0 in I is mapped diffeomorphically into the unstable manifold W_p^u. There is a neighborhood of 1 in I whose image by h is included in W_q^s or W_q^u. Hence we have $\frac{dg}{dt}(1) \neq 1$.

We see that $0 < \frac{dg}{dt}(1) < 1$ since g has no fixed points except the boundary points 0 and 1. Therefore there is a neighborhood of 1 in I which is mapped into the stable manifold W_q^s of q. As manifolds W_p^u and W_q^s are invariant by f and that g is a diffeomorphism having no fixed points except 0 and 1, the image $h(I-\{1\})$ is included totally in W_p^u and $h(I-\{0\})$ is included in W_q^s.

By a theorem obtained by the author [1], there exist analytic mappings $\varphi_p : E_p^u \to W_p^u$ and $\varphi_q : E_q^s \to W_q^s$ satisfying the following conditions.

i) $\varphi_p(0) = p$, $\varphi_q(0) = q$

ii) $d\varphi_p(0) : E_p^u \to T_pR^2$ and $d\varphi_q(0) : E_q^s \to T_qR^2$ are inclusion mappings.

iii) $\varphi_p(E_p^u) = W_p^u$, $\varphi_q(E_q^s) = W_q^s$.

iv) φ_p and φ_q are injective immersions.

v) $f(\varphi_p(\xi)) = \varphi_p(\alpha\xi)$, $f^{-1}(\varphi_q(\xi)) = \varphi_q(\beta^{-1}\xi)$

where α (resp. β) denotes the eigenvalue of df_p (resp. df_q) associated to the eigenspace E_p^u (resp. E_q^s).

Moreover, in our case where f can be extended to an automorphism of C^2, these mappings φ_p and φ_q can be extended holomorphically to entire mappings

$$\Phi_p : C \to C^2 \quad \text{and} \quad \Phi_q : C \to C^2 ,$$

i.e., Φ_p and Φ_q are holomorphic mappings defined globally on C and that $\Phi_p\big|_R = \varphi_p$ and $\Phi_q\big|_R = \varphi_q$. Mappings Φ_p and Φ_q are injective immersions, too. Note that the image $\Phi_p(C)$ does not contain q and $\Phi_q(C)$ does not contain p. By the assumption of the existence of a saddle connection $h : I \to R^2$, we conclude that each of two complex curves $\Phi_p(C)$ and $\Phi_q(C)$ includes the image $h(I-\{0,1\})$. On the other hand, the intersection $\Phi_p(C) \cap \Phi_q(C)$ of two complex curves must be of complex dimension one or zero. Now that it is at least of real dimension one, it follows that its complex dimension is one at least in the neighborhood of the image $h(I-\{0,1\})$.

Let U and V denote the inverse image of this intersection by Φ_p and Φ_q respectively. We have a holomorphic function $\psi : U \to V$ defined by $\psi = \Phi_q^{-1} \circ \Phi_p$. Mappings Φ_p and Φ_q being injective immersions, ψ is a bijective mapping. Mapping ψ is also an isomorphism. Let M denote the one dimensional complex manifold defined by two holomorphic charts C_p and C_q with the coordinate transformation map $\psi : U \to V$. More precisely, manifold M is composed of two copies C_p and C_q of the complex plane C of which two points $z \in C_p$ and $w \in C_q$ are identified if $w = \psi(z)$. By an elementary argument using the uniqueness of analytic continuation, it is easily verified that this manifold M is Hausdorff. Manifold M is immersible in C^2. In fact, for a point $z \in C_p$ or $w \in C_q$, define its image by $\Phi_p(z)$ or $\Phi_q(w)$. We denote this mapping by $\sigma : M \to C^2$.

Define an entire mapping $\gamma : C \to M$ by $\gamma(z) = z \in C_p \subset M$. Mapping γ is holomorphic on C. Mapping γ is not surjective since the origin of C_q is not contained in its image. Let $\pi : \tilde{M} \to M$ be the universal covering of M. As the complex plane C is simply connected, mapping $\gamma : C \to M$ can be lifted into mapping $\tilde{\gamma} : C \to \tilde{M}$. By the theorem of Koebe on the classification of simply connected one dimensional complex manifolds, $\tilde{M}$ is isomorphic to the complex plane C, the interior of the unit disc $D = \{z \in C \mid |z| < 1\}$ or the complex projective space CP. As there exists an entire function $\tilde{\gamma} : C \to \tilde{M}$ which is injective and not surjective, $\tilde{M}$ cannot be isomorphic to C nor to D. Finally if $\tilde{M}$ is isomorphic to CP, the composed map $\sigma \circ \pi \circ \tilde{\gamma} = \sigma \circ \gamma : C \to C^2$ will be entire. But the projective space CP is compact hence its image by σ is bounded. Therefore $\sigma \circ \gamma$ is a constant map, which contradicts to the injectivity of the composed map $\sigma \circ \gamma = \Phi_p$.

Now consider the case where p and q are identical. The point $p = q$ is a saddle point so that the unstable manifold W^u_p and W^s_q intersect at the point transversally. We bolw up the space C^2 at p. Then we have a complex manifold of dimension two. We denote this manifold by H. Let $\rho : H \to C^2$ be the canonical projection map. Mapping ρ is holomorphic and bijective on $\rho^{-1}(C^2 - \{p\})$. The inverse image $\rho^{-1}(p)$ is isomorphic to the complex projective space CP. Eigenspaces E^u_p and E^s_q , which are tangent to W^u_p and W^s_q respectively at p, define points p' and q' respectively in $\rho^{-1}(p)$. The transversality of W^u_p and W^s_q implies that p' and q' are distinct in H. In place of entire mappings Φ_p and Φ_q , take modified mappings $\Psi_p : C \to H$ and $\Psi_q : C \to H$ defined as follows :

$$\Psi_p(0) = p' , \quad \Psi_q(0) = q'$$

$$\Psi_p(\xi) = \rho^{-1} \circ \Phi_p(\xi) , \quad \Psi_q(\xi) = \rho^{-1} \circ \Phi_q(\xi) \quad \text{for } \xi \neq 0.$$

By the same arguement, we will have a contradiction. We proved our theorem also for saddle connection curve whose starting point and ending point are same.

Remark We used the supposition of the existence of saddle connection only to derive that the complex manifolds $\Phi_p(C)$ and $\Phi_q(C)$ intersect in a certain portion of complex dimension one. We have the following generalization of our theorem.

Theorem 2 Let V be a finite-dimensional complex manifold. Assume V can be immersed holomorphically in finite-dimensional complex vector sapce C^N. Let $f : V \to V$ be an automorphism of V and let p and q be hyperbolic fixed points. We assume that the unstable manifold of p and the stable manifold of q of dynamical system (V,f) are one-dimensional. Then there exists no embedding $h : I \to V$ of the unit interval into V whose image $h(I)$ is included in the unstable manifold of p and the stable manifold of q simultaneously.

The proof is similar to that for theorem 1.

We employ a theorem of the author [1],[2] for the case of automorphisms of complex manifolds in order to obtain the entire mappings parametrizing unstable manifolds and stable manifolds.
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REFERENCES

[1] S.Ushiki : Analytic expressions of unstable manifolds, Proc. Jap. Acad., vol.56, ser. A, p.239-244,1980.

[2] S.Ushiki : Unstable manifolds of analytic dynamical systems. (to appear in Kyoto J. of Math.)

[3] S.Ushiki : Sur les liaisons-cols des systèmes dynamiques analytiques, Comptes Rendus Acad. Sci. Paris, t. 291, Sér. A, p.447-449, 1980.

[4] M.Yamaguti et S.Ushiki : Discrétisation et chaos, Comptes Rendus Acad. Sci. Paris, t.290, N°14, p.637-640, 1980.

[5] M.Yamaguti and S.Ushiki : Chaos in numerical analysis of ordinary differential equations. (to appear in PHYSICA D)

[6] S.Ushiki : Central difference scheme and chaos. (to appear in PHYSICA D)