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Chaos in one-dimensional dynamical systems

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In this note we would like to mention several results about the chaos in one-dimensional dynamical systems obtained recently in Japan. We will divide these results into the following three classes, (1) chaos and Sarkovskii's theorem, (2) statistical mechanics and chaos and (3) random systems. We will state the results only in this note, so please refer to each paper for futher particulars.

§1. Chaos and Sarkovskii's theorem

In [1], we obtain almost complete results for unimodal linear transformations. Unimodal linear transformation is the continuous map on $[0,1]$ into itself with one extremum point and monotone on each side of extremum point. Except some trivial cases, this map reduces to the function $f_{a,b}(x)$ defined by

$$f_{a,b}(x) = \begin{cases} ax + \frac{a+b-ab}{b} & 0 \leqq x \leqq 1-\frac{1}{b} \\ -b(x-1) & 1-\frac{1}{b} < x \leqq 1 \end{cases}$$

with parameter $(a,b) \in D = \{(a,b);\ b>1,\ ab>1,\ a+b \geqq ab\}$. And according to this family of functions we have the following results.

(1) In the case of $D_k^{(1)} = \{(a,b);\ a<1,\ 1+a^{-1}+\cdots+a^{-(k-1)} < b \leqq 1+ +a^{-1}+\cdots+a^{-k},\ a^k b \leqq 1\}$, there exists a stable periodic orbit with period k+1 and almost all orbit approaches this periodic orbit. And so in this case $f_{a,b}$ has no absolutely continuous invariant measure. We call this case "window".

(2) In the case of $D_k^{(2)} = \{(a,b);\ a<1,\ 1+a^{-1}+\cdots+a^{-(k-1)} < b \leqq 1+a^{-1}+\cdots+a^{-k},\ a^k b>1,\ a+b \geqq a^k b^2\}$, $f_{a,b}$ has an absolutely continuous invariant measure μ whose support is union of disjoint intervals $\bigcup_{i=0}^{k} J_i$. And it satisfies that $f_{a,b}(J_i)=J_{i+1}$ (mod(k+1)) and $f_{a,b}^{(k+1)}|_{J_i}$ is weakly mixing with respect to $\mu|_{J_i}$. We call this case "islands". Here, $f_{a,b}^{(k+1)}|_{J_i}$ is equivalent to the case (3).

(3) Except the cases (1), (2) and $D_0 = \{(a,b);\ \frac{a+b-ab}{b} \geqq \frac{b}{b+1}\}$ (D_0 is the case of $2^n \times$odd(>1)-period case), $f_{a,b}$ has an absolutely continuous invariant measure whose support is $[0,1]$, and $f_{a,b}$ is weak-Bernoulli with respect to this measure. In the cases (2) and (3), the density function of invariant measure is given by

$$h_{a,b}(x) = \sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^{N(n)} \left(-\frac{1}{b}\right)^{n-N(n)} I_{[f_{a,b}^n(0),1]}(x)$$

where $N(n) = \#\{0 \leqq i \leqq n-1;\ f_{a,b}^i(0) \leqq 1-\frac{1}{b}\}$.

In [2], we define the type of periodic orbit of unimodal transformation on an interval and extend Sarkovskii's theorem. Let f be a continuous map on $[0,1]$ into itself satisfying $f(c)=1$, $f(1)=0$ for some $0<c<1$ and being monotone increasing on $[0,c]$ and monotone decreasing on $[c,1]$. The type of periodic point x of f is $(l_1,r_1)(l_2,r_2)\cdots(l_n,r_n)$ if x is periodic point of period $l_1+r_1+\cdots+l_n+r_n$ and satisfies that $f^i(x) \in [0,c]$ if and only if $0 \leqq i < l_1$, $l_1+r_1 \leqq i < l_1+r_1+l_2$, $\cdots$, $l_1+r_1+\cdots+l_{n-1}+r_{n-1} \leqq i < l_1+r_1+\cdots+l_{n-1}+r_{n-1}+l_n$. In the set of all type of periodic orbit we introduce natural order which corresponds to the order of the minimal point among the orbit, then we obtain the following extension of Sarkovskii's theorem. If f has a periodic point of type t, then f has all periodic point of type greater than t. Using this result, we can calculate the topological entropy of f.

In [3], we treat maps of the circle. Let f be a continuous map of the circle into itself with mapping degree d. Then we have the following results.

(1) In the case that d=0 or -1, assume that f has a periodic point of period n, an odd integer $\geqq 3$, then we have

$$h(f) \geqq \log \sigma_n$$

$$\liminf \frac{1}{k} \log p_k(f) \geqq \log \sigma_n ,$$

where h(f) is the topological entropy of f, $p_k(f)$ is the number of periodic points of f of period k and σ_n is the unique positive root of the equation $t^n - t^{n-1} - 1 = 0$.

(2) In the case that d=1, assume that f has both fixed point and a periodic point of period n, an odd prime integer, then we have the same inequalities as in (1).

(3) In the case that $|d| \geqq 2$, we have

$$h(f) \geqq \log|d|$$

$$\liminf \frac{1}{k} \log p_k(f) \geqq \log|d| .$$

(4) Making use of the formula $h(f^m) = m \cdot h(f)$ and above results, we can show that if f has a fixed point and h(f)=0, then the period of any periodic point of f is a power of 2.

In [4], we give the definition of pseudo-Markov transformations.

The map f on an interval I into itself is pseudo-Markov if there exists a family of intervals $\{I_u;\ u \in W\}$ which satisfies (1) W is a set of words of some Markov subshift, (2) I_u is a closed subinterval of I ($\neq \phi$), (3) f maps $I_{a_1,a_2,\cdots,a_n}$ onto $I_{a_2,\cdots,a_n}$ and (4) $I_{a_1,\cdots,a_n} \subset I_{a_1,\cdots,a_m}$ for each $m < n$. And we can show that pseudo-Markov transformation has an Markov invariant measure.

In [5], we give the definition of formal chaos. We call that f shows formal chaos if there exist disjoint closed intervals I_1, I_2 and a natural number n which satisfy

$$f^n I_1 \cap f^n I_2 \supset I_1 \cup I_2 .$$

This definition is equivalent to the condition that f is pseudo-Markov and its representation to a Markov subshift is mixing. We can show that if f shows formal chaos then it also shows Li-Yorke's chaos.

In [6], we give the realization (X_f, σ), in general sense, of continuous map f on an interval into itself, and using this realization we obtain the relation

$$h(f) = \sup \{ h(f_\tau);\ \tau = \tau(C),\ C\text{:cycle of } f \}$$

where f_τ is a map obtained by connecting points $(x_i, f(x_i))$, $x_i \in C$ by straight line. We also obtain the relation

$$h(f) = \limsup \frac{1}{n} \log \# \{ \text{cycle of } f \text{ with period } n \} .$$

In [7], we give the definition of the shift with orbit base. Let (X, σ) be a subshift of $(A^{\mathbb{N}}, \sigma)$. We call X a shift with orbit basis B ($B \subset W(X)$) if there exists $V \subset B \times B$ which satisfies

(1) each $\omega \in X$ can be uniquely represented as

$$\omega = \sigma^i b_0 \cdot b_1 \cdot b_2 \cdots \quad (0 \leqq i < |b_0|)$$

for some $b_0, b_1, b_2, \cdots \in B$ with $(b_{n-1}, b_n) \in V$ and (2) each $\omega \in A^{\mathbb{N}}$ which is represented as in (1) is contained in X. We can show that for any continuous map on an interval into itself, X_f is a shift with orbit basis B for some B.

§ 2. Statistical mechanics and chaos

In [8], we show the relation between one-dimensional dynamical system and statistical mechanics, and show that the value of free energy P determines the existence of chaos. The map f on an interval into itself shows observable chaos if f has an observable invariant measure μ, that is,

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{f^n(x)} \longrightarrow \mu \quad \text{for all } x \in A$$

for some set A with positive Lebesgue measure, and f is mixing with respect to μ. From our experiences we can conjecture that this measure μ is absolutely continuous with respect to the Lebesgue measure, and its density $\varphi(x)$ satisfies

$$\varphi(x) = \mathcal{L}_f \varphi(x)$$

$$= \sum_{y \in f^{-1}(x)} \frac{\varphi(y)}{|f'(y)|} .$$

It is known that the free energy P is given by

$$P = \limsup \frac{1}{n} \log Q_n$$

where $Q_n = \mathrm{Tr} \mathcal{L}_f n = \sum_{z=f^n(z)} \frac{1}{|(f^n)'(z)|}$, and we can state the following results (some of them are conjecture). (1) P satisfies the variational principle

$$P = \sup_{\mu : f\text{-inv.}} \left(h_\mu(f) - \int \log|f'| \, d\mu \right) .$$

(2) If f has an attracting periodic orbit, that is, the case of "window", then $P > 0$.

(3) If $P < 0$, then f has no absolutely continuous invariant measure.

(4) If $P=0$, then f has an absolutely continuous invariant measure and f shows observable chaos.

In [9], we give mathematical proof of some of the above results precisely in the case of piecewise C^1-function f with $\inf|f'| > 0$. (Also see [10] and [11])

§3. Random systems

In [12] and [13], we treat the following system (introducing randomness to the parameter of transformation). Let $\{f_\alpha , \alpha \in A\}$ be a one parameter family of maps and let $\{X_n\}$ be a sequence of independent and identically distributed A-valued random variables. And we define the orbit $\{x_n, n \geq 0\}$ starting from x_0 by

$$x_1 = f_{X_1}(x_0),\ x_2 = f_{X_2}(x_1), \cdots ,\ x_n = f_{X_n}(x_{n-1}), \cdots .$$

In [12] we consider the case that f_α is unimodal linear transformation

$$f_\alpha(x) = \begin{cases} \alpha x & 0 \leq x \leq \frac{1}{2} \\ -\alpha(x-1) & \frac{1}{2} < x \leq 1 , \end{cases}$$

$A = \{a, b\}$ ($0 < a \leq b \leq 2$) and $P\{X_n = a\} = P\{X_n = b\} = \frac{1}{2}$. We can represent this system into skew product transformation T on $[0,1] \times \{a,b\}^{\mathbb{N}}$ defined by

$$T(x,y) = (f_{y_1}(x), \sigma y)$$

where y_1 is the first coordinate of y and σ is the shift operator in $\{a,b\}^{\mathbb{N}}$. Then we obtain the following results.

(1) In the case that $ab \leqq 1$, T has no absolutely continuous invariant probability measure.

(2) In the case that $ab > 1$, T has an absolutely continuous invariant probability measure with y-independent density function h(x) given by

$$h(x) = \frac{1}{C} \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{y_1, \cdots, y_n} \frac{S(y_1, \cdots, y_{n-1})}{y_1 \cdots y_n} I_{[0, f_{y_n} \cdots f_{y_1}(\frac{1}{2})]}(x)$$

where $S(y_1, \cdots, y_{n-1}) = (-1)^{\# \{0 \leqq i \leqq n-1;\ f_{y_i} \cdots f_{y_1}(\frac{1}{2}) > \frac{1}{2}\}}$.

And so, for almost all ω and almost all x_0,

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_n} \longrightarrow h(x)dx .$$

(3) In the case that $ab > 1$, T is ergodic with respect to this measure. Moreover, T is exact under some additional condition. It is easily checked that T is exact if f_b is exact, but T may be exact even if f_a and f_b are not exact, for example, the case that $a=1$ and $1 < b < \sqrt{2}$.

In [13], we consider more general case of random system. Let A, the set of parameter, be an interval and the distribution of X_n be ν (with continuous density on A). We represent this system into skew product transformation T on $I \times A^{\mathbb{N}}$ defined by

$$T(x, \omega) = (f_{\omega_1}(x), \sigma\omega).$$

Let Π be a Markov process with transition probability $p(y|x)dy$

where the density function $p(y|x)$ is given by

$$\int \varphi(y)p(y|x)dy = \int \varphi(f_a(x))d\nu(a) .$$

Then we obtain the following results.

(1) If ν is Π-invariant measure, then ν is absolutely continuous measure.

(2) If ν is Π-ergodic, then $\nu \times \mathbb{b}$ is T-ergodic, where $\mathbb{b}$ is the product measure of ν on $A^{\mathbb{N}}$. Moreover, in this case, ν is loosely exact, that is, there exist a natural number n and Π^n-exact measure $\nu^{(n)}$ which satisfy

$$\nu = \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}^{k*}(\nu^{(n)})$$

where $\mathcal{P}^*$ is defined from $p(y|x)$.

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