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The Onsager-Machlup functions for diffusion processes

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For the n-dimensional Wiener measure P^W, we often write formally as

$$P^W(\cdot) = \int_{\cdot} \text{const.} \exp[-\frac{1}{2}\int_0^T \|\dot{\phi}_t\|^2 dt]\mathcal{D}(\phi)$$

the functional $\exp[-\frac{1}{2}\int_0^T \|\dot{\phi}_t\|^2 dt]$ representing thus an ideal density with respect to a fictitious uniform measure on the space of all continuous paths $\phi=(\phi_t) : [0,T] \to R^n$. Our problem is concerned with defining such densities for general diffusion processes. If this density is of the form

$$\text{const.} \exp[-\int_0^T L(\phi_t, \dot{\phi}_t) dt]$$

with a function L on the tangent bundle, L is naturally regarded as Lagrangian associated with the diffusion process. In physical literatures, such a function is called the Onsager-Machlup function of the diffusion process, cf. [2],[4],[6],[7].

As a mathematical problem, it has the following two aspects. First, we would like to find a function L such that the transition probability density $p(t,x,y)$ of the diffusion process admits a representation

$$p(t,x,y) = \int_{\{\phi:\phi_0=x,\phi_t=y\}} \text{const.}\ \exp[\ -\int_0^t L(\phi_s,\dot{\phi}_s)ds]\ \mathcal{D}(\phi)$$

by a so-called <u>path integral</u> or <u>continuum integral</u>. This problem is closely related to the asymptotic evaluation of $p(t,x,y)$ as $t \downarrow 0$ and was studied in , e.g.,[1],[4]. It seems, however, that a rigorous justification is still open.

The second aspect is that, since a diffusion process is a well-defined mathematical object as a probability measure on the space of paths, L might be obtained by evaluating asymptotically the measure of those paths which lie in a small tube around a given smooth curve. Such an evaluation was attempted by Stratonovich [8] where the functional thus obtained is called the <u>probability functional</u> of the diffusion process. Let us rigorously formulate the problem. Let (X_t,P_x) be a locally conservative, non-singular diffusion process on a manifold M ("locally conservative" means that the infinitesimal generator of the diffusion does not contain the 0-th order term and "non-singular" means that it is strictly elliptic: the case of the non-zero 0-th order term can be treated by the Feynman-Kac formula). We assume that the coefficients of the infinitesimal generator are smooth. Then a Riemannian structure is naturally induced by the diffusion coefficients so that the generator is $\frac{1}{2}\Delta_M + b$ where Δ_M is the Laplace-Beltrami operator and b is a vector field and an intrinsic metric defining the tube should be the Riemannian distance $\rho(x,y)$. We may thus formulate the problem as follows: let M be a Riemannian manifold of the dimension n

, (X_t, P_x) be the diffusion process on M with the infinitesimal generator $\frac{1}{2}\Delta_M + b$ and $\phi=(\phi_t) : [0,T] \to M$ be a smooth curve. Find an asymptotic formula for the probability

$$P_{\phi_0}(\rho(X_t,\phi_t) < \varepsilon \quad \text{for all} \quad t \in [0,T])$$

as $\varepsilon \downarrow 0$.

An answer is given in the following theorem.

THEOREM

$$P_{\phi_0}(\rho(X_t,\phi_t) < \varepsilon \quad \underline{\text{for all}} \quad t \in [0,T])$$

$$\sim C \exp(-\lambda_1 T/\varepsilon^2) \exp[-\int_0^T L(\phi_t,\dot{\phi}_t)dt] \qquad \underline{\text{as}} \quad \varepsilon \downarrow 0,$$

where L is a function on the tangent bundle TM defined by

$$L(x,\dot{x}) = \frac{1}{2}\| b(x) - \dot{x}\|^2 + \frac{1}{2}\,\text{div}\, b(x) - \frac{1}{12} R(x).$$

Here $\| \ \|$ is the Riemannian norm in the tangent space $T_x(M)$, $R(x)$ is the scalar curvature (i.e. the trace of the Ricci curvature), $C = \psi_1(0)\int_D \psi_1(x)dx$ and $(\psi_m(x),\lambda_m)_{m=1,2,\dots}$ is the eigen-system for $-\frac{1}{2}\Delta_{R^n}$ (Δ_{R^n} : the Laplacian in R^n) in the unit ball D of R^n with Dirichlet's boundary condition.

The above theorem was conjectured and verified to hold in Einstein spaces by Y.Takahashi [9] through probabilistic method using e.g. Girsanov's theorem, Levy's stochastic area integrals and stochastic Stokes's theorem. The final proof was given in [10] by accomplishing Takahashi's probabilistic approach and also in [3] by using purely analytical method.

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