数理解析研究所講究録
第 445 巻 1981 年 29-40

On approximative movability

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0. Introduction. Recently many generalizations of the notion of absolute neighborhood retract(ANR) have studyed. They are approximative absolute neighborhood retact (AANR) of Clapp [3] and NE-sets of Borsuk [2] for compact metric spaces. Mardešić [4] introduced the notion of approximative polyhedra (AP) for arbitrary spaces, and he proved that these there notions are equivalent for compact metric spaces.

In this note we shall introduce a new notion of approximative movability for arbitrary spaces, and show that this notion is equivalent to the above there notions for compact metric spaces. Also we give a characterization of AP for arbitary spaces by using resolutions.

Recently the author introduces approximative shape theory. However we do not discuss on it, because it needs many pages for it. The detailed description of approximative shape theory will be appeared in elsewhere.

1. Preliminaries. All spaces and maps are topological spaces and continous maps,respectively. By a polyhedron we mean the realization of a simplicial complex with CW-topology. By an ANR we mean an absolute neighborhood retract for metric spaces. In this note, coverings are always normal open coverings. Cov(X) denotes the collection of all coverings of a space X. Let $\underline{U}$ and $\underline{V}$ be coverings of X. We say that $\underline{U}$ is a refinement of $\underline{V}$, in notation $\underline{U} \leqq \underline{V}$, provided that for each $U \in \underline{U}$ there exists a $V \in \underline{V}$ with $U \subseteq V$. For a subset A of X, put $st(A,\underline{U}) = \cup \{U \in \underline{U};\ A \cap U \neq \emptyset\}$. Put $st(\underline{U}) = \{st(U,\underline{U})\ ;\ U \in \underline{U}\}$, then $st(\underline{U})$ forms a covering of X. Inductively we define $st^n(\underline{U})$ by $st(st^{n-1}(\underline{U}))$ for each integer n. Note that any $\underline{U} \in Cov(X)$ has a $\underline{V} \in Cov(X)$ with $st^n(\underline{V}) \leqq \underline{U}$ for each integer n. This fact follows from the definition of normal coverings. Let $f,g : X \longrightarrow Y$ be maps and $\underline{V} \in Cov(Y)$. Then $f^{-1}(\underline{V}) = \{f^{-1}(V)\ ;\ V \in \underline{V}\}$ forms a covering of X. We say that f and g are $\underline{V}$-near, in notation $(f, g) \leqq \underline{V}$, provided that for each $x \in X$ there exists a $V \in \underline{V}$ with $f(x), g(x) \in V$.

Definition 1 (Mardešić[4]). A space X is an approximative polyhedron provided that for each $\underline{U} \in Cov(X)$, there exist maps $f: X \longrightarrow P$ and $g: P \longrightarrow X$ with $(gf, 1_X) \leqq \underline{U}$, where P is a polyhedron.

It is easy to show that all ANRs are approximative polyhedra. In [4] he proved that every LC^{n-1} paracompact space X with dim $X \leqq n$ is an approximative polyhedron.

Let $\underline{X} = \{X_a, p_{a'a}, A\}$ be an inverse system of spaces. Let $\underline{p} = \{p_a;\ a \in A\} : X \longrightarrow \underline{X}$ be a system of maps, that is, $p_a = p_{a'a}p_{a'}$ for each $a' \geq a$. We say that $\underline{p}$ is a resolution of a space X provided that it satisfies the following two conditions:

(R1) Let P be an approximative polyhedron, $\underline{V}$ be a covering of P, and $f: X \longrightarrow P$ be a map. Then there exist an $a \in A$ and a map $f_a : X_a \longrightarrow P$ with $(f_a p_a, f) \leq \underline{V}$.

(R2) Let P be an approximative polyhedron and $\underline{V}$ be a covering of P. Then there exists a $\underline{V}' \in \mathrm{Cov}(P)$ with the following property; if $a \in A$ and $f,g : X_a \longrightarrow P$ are maps with $(fp_a, gp_a) \leq \underline{V}'$, then there exists an $a' \in A$ such that $a' \geq a$ and $(fp_{a'a}, gp_{a'a}) \leq \underline{V}$.

We say that $\underline{p}$ is an AP-resolution (polyhedral-resolution, ANR-resolution) of X provided that all X_a are AP (polyhedra, ANR, respectively). These notion are introduced by Mardešić [4], and he showed that

Lemma 1. Any space has an AP-resolution (polyhedral, ANR-resolutions).

2. Approximatively internally movable spaces. In this section we introduce the notion of approximatively internally movable spaces. Let X be a space , and $\underline{p} = \{p_a;\ a \in A\}\ X \longrightarrow \underline{X} = \{X_a, p_{a'a}, A\}$ be an AP-resolution of X.

Definition 2. An AP-resolution $\underline{p} : X \longrightarrow \underline{X}$ is approximatively internally movable provided that for each $a \in A$ and for each $\underline{U} \in Cov(X_a)$, there exist an $a' \in A$ with $a' \geq a$ and a map $r: X_{a'} \longrightarrow X$ such that $(p_a r, p_{a'a}) \leq \underline{U}$.

We show the following prperty of approximatively internal movability.

Proposition 1. The prperty of approximatively internal movability does not depend on AP-resolutions.

Proof. Let $\underline{p}: X \longrightarrow \underline{X}$ and $\underline{q} = \{q_b : b \in B\} : X \longrightarrow \underline{Y} = \{Y_b, q_{b'b}, B\}$ be AP-resolutions of a space X. We assume that $\underline{p}$ satisfies the condition of Definition 2. We show that $\underline{q}$ also satisfies the same condition.

Take any $b \in B$ and any covering $\underline{V}$ of Y_b. By (R2), there exists $\underline{V}_1 \in Cov(Y_b)$ satisfying the condition (R2) for $\underline{q}$ and $\underline{V}$. Let $\underline{V}_2$ be a covering of Y_b with $st^2(\underline{V}_2) \leq \underline{V}_1$. By using (R1) for q_b and $\underline{V}_2$, there exist an $a \in A$ and a map $h: X_a \longrightarrow Y_b$ such that

(1) $(hp_a, q_b) \leq \underline{V}_2$.

Since $\underline{p}$ satisfies the condition of Definition 2, there exist an $a' \in A$ with $a' \geq a$ and a map $r: X_{a'} \longrightarrow X$ such that

(2) $(p_a r, p_{a'a}) \leq h^{-1}(\underline{V}_2)$.

From (2) we have that

(3) $(hp_a r, hp_{a'a}) \leq \underline{V}_2$.

By using (R1) for $p_{a'}$ and $(hp_{a'a})^{-1}(\underline{V}_2)$, there exist a $b' \in B$ with $b' \geq b$ and a map $k: Y_{b'} \longrightarrow X_{a'}$ such that

(4) $(kq_{b'} , p_{a'}) \leqq (hp_{a'a})^{-1}(\underline{V}_2)$.

From (4) we have that

(5) $(hp_{a'a}kq_{b'} , hp_a) \leqq \underline{V}_2$.

From (3) we have that

(6) $(hp_a rkq_{b'} , hp_{a'a}kq_{b'}) \leqq \underline{V}_2$.

From (1) we have that

(7) $(hp_a rkq_{b'} , q_b rkq_{b'}) \leqq \underline{V}_2$.

From (1), (5), (6) and (7) we have that

(8) $(q_b rkq_{b'} , q_{b'b}q_{b'}) \leqq \underline{V}_1$.

By choosing of $\underline{V}_1$ and (8), there exists a $b'' \in B$ with $b'' \geqq b'$ such that

(9) $(q_b rkq_{b''b'} , q_{b''b}) \leqq \underline{V}$.

Hence (9) means that $rkq_{b''b'} : Y_{b''} \longrightarrow X$ satisfies the required condition. Then $\underline{q}$ satisfies the condition of Definition 2. This completes the proof of Proposition 1.

Since we have Proposition 1 we can define the following property:

Definition 3. A space X is approximatively internally movable provided that any AP-resolution of X is approximatively internally movable.

In shape theory the notion of internal movability was introduced by Bogatyĭ [/]. We characterize AP as follows:

Theorem 1. A space X is an approximative polyhedron if and only if it is approximatively internally movable.

Proof. First we assume that X is an approximative polyhedron. We consider the AP-resolution $\underline{p} = \{1_X\} : X \longrightarrow X$, that is, rudimentary resolution of X. Since X is an approximative polyhedron $\underline{p}$ is an AP-resolution of X and obviously $\underline{p}$ satisfies the condition of Definition 2. Hence by Proposition 1, any AP-resolution of X is approximatively internally movable. Therefore X is approximatively internally movable.

Next, we assume that X is approximatively internally movable. By Lemma 1, there exists a polyhedral resolution $\underline{p} = \{p_a \, ; a \in A\} : X \longrightarrow \underline{X} = \{X_a, p_{a'a}, A\}$ of X. Since X is approximatively internally movable, $\underline{p}$ is approximatively internally movable. Take any covering $\underline{U}$ of X. By Theorem 5 of Mardešić [4], there exist an $a \in A$ and a $\underline{V} \in Cov(X_a)$ with

(1) $p_a^{-1}(\underline{V}) \leqq \underline{U}$.

Since $\underline{p}$ is approximatively internally movable, there exist an $a_1 \in A$ with $a_1 \geqq a$ and a map $r: X_{a_1} \longrightarrow X$ such that

(2) $(p_a r, p_{a_1 a}) \leqq \underline{V}$.

From (2) we have that

(3) $(p_a r p_{a_1}, p_a) \leqq \underline{V}$.

Take any $x \in X$. From (3) there exists a $V \in \underline{V}$ such that

(4) $p_a r p_{a_1}(x), p_a(x) \in V$.

From (4) we have that

(5) $r p_{a_1}(x), x \in p_a^{-1}(V)$.

From (1) there exists a $U \in \underline{U}$ with $p_a^{-1}(V) \subset U$. Hence from (5) we have

(6) $rp_{a_1}(x), x \in U$.

(6) means that

(7) $(rp_{a_1}, 1_X) \leq \underline{U}$,

that is, maps $p_{a_1} : X \longrightarrow X_{a_1}$ and $r : X_{a_1} \longrightarrow X$ satisfies the condition of Definition 1. Hence X is an approximative polyhedron. This completes the proof of Theorem 1.

3. Approximatively movable spaces. In this section we introduce the notion of approximative movability. Let X be a space. Let $\underline{p} = \{p_a : a \in A\} : X \longrightarrow \underline{X} = \{X_a, p_{a'a}, A\}$ be an AP-resolution of X.

Definition 4. We say that $\underline{p}$ is approximatively movable provided that for each $a \in A$ and for each $\underline{U} \in Cov(X_a)$, there exists an $a_0 \in A$ with $a_0 \geqq a$ such that for each $a_1 \in A$ with $a_1 \geqq a$, there exists a map $r : X_{a_0} \longrightarrow X_{a_1}$ satisfying $(p_{a_1 a} r, p_{a_0 a}) \leqq \underline{U}$.

We show the following property of approximative movability.

Proposition 2. The property of approximative movability does not depend on AP-resolutions.

Proof. Let $\underline{p} : X \longrightarrow \underline{X}$ and $\underline{q} = \{q_b, b \in B\} : X \longrightarrow \underline{Y} = \{Y_b, q_{b'b}, B\}$ be AP-resolutions of a space X. We assume that $\underline{p}$ is approximatively movable. We show that $\underline{q}$ is also approximatively movable.

Take any $b \in B$ and any covering $\underline{U}$ of Y_b. Let $\underline{U}_1$ be a covering of

Y_b with $st(\underline{U}_1) \leqq \underline{U}$. By (R2) there exists a $\underline{U}_2 \in Cov(Y_b)$ with $\underline{U}_2 \leqq \underline{U}_1$ satisfying the property of (R2) for $\underline{q}$ and $\underline{U}_1$. By (R2) there exists a $\underline{V}_2 \in Cov(Y_b)$ with $\underline{V}_2 \leqq \underline{U}_1$ satisfying the property of (R2) for $\underline{p}$ and $\underline{U}_1$. Let $\underline{U}_3$ be a covering of Y_b with $st(\underline{U}_3) \leqq \underline{U}_2$ and $st(\underline{U}_3) \leqq \underline{V}_2$. By (R1) for $\underline{p}$, there exist an $a \in A$ and a map $h : X_a \longrightarrow Y_b$ such that

$$(1) \quad (hp_a, q_b) \leqq \underline{U}_3.$$

Since $\underline{p}$ is approximatively movable, there exists an $a_1 \in A$ with $a_1 \geqq a$ satisfying the property of Definition 4 for a and $h^{-1}(\underline{U}_3)$. By (R1) for $\underline{q}$, there exist a $b_1 \in B$ with $b_1 \geqq b$ and a map $k : Y_{b_1} \longrightarrow X_{a_1}$ such that

$$(2) \quad (kq_{b_1}, p_{a_1}) \leqq (hp_{a_1a})^{-1}(\underline{U}_3).$$

From (2) we have that

$$(3) \quad (hp_{a_1a}kq_{b_1}, hp_a) \leqq \underline{U}_3.$$

From (1) and (3) we have that

$$(4) \quad (hp_{a_1a}kq_{b_1}, q_{b_1b}q_{b_1}) \leqq \underline{U}_2.$$

By choosing of $\underline{U}_2$ and (4), there exists an $b_2 \in B$ with $b_2 \geqq b_1$ such that

$$(5) \quad (hp_{a_1a}kq_{b_2b_1}, q_{b_2b}) \leqq \underline{U}_1.$$

We show that b_2 is the required one. Take any $b' \in B$ with $b' \geqq b$. By (R1), there exist an $a' \in A$ with $a' \geqq a$ and a map $m: X_{a'} \longrightarrow Y_{b'}$ such that

$$(6) \quad (mp_{a'}, q_{b'}) \leqq q_{b'b}^{-1}(\underline{U}_3).$$

From (6) we have that

(7) $(q_{b'b}mp_{a'} , q_b) \leqq \underline{U}_3$.

From (1) and (7), we have that

(8) $(q_{b'b}mp_{a'} , hp_{a'a}p_{a'}) \leqq \underline{V}_2$.

By choosing of $\underline{V}_2$, there exists an $a'' \in A$ with $a'' \geqq a', a_1$ such that

(9) $(q_{b'b}mp_{a''a'} , hp_{a''a}) \leqq \underline{U}_1$.

By choosing of a_1, there exists a map $r: X_{a_1} \longrightarrow X_{a''}$ such that

(10) $(p_{a''a}r , p_{a_1a}) \leqq h^{-1}(\underline{U}_3)$.

From (10) we have that

(11) $(hp_{a''a}r , hp_{a_1a}) \leqq \underline{U}_3$.

From (11) we have that

(12) $(hp_{a''a}rkq_{b_2b_1} , hp_{a_1a}kq_{b_2b_1}) \leqq \underline{U}_3 \leqq \underline{U}_1$.

From (9) we have that

(13) $(q_{b'b}mp_{a''a'}rkq_{b_2b_1} , hp_{a''a}rkq_{b_2b_1}) \leqq \underline{U}_1$.

From (5), (12) and (13) we have that

(14) $(q_{b'b}mp_{a''a'}rkq_{b_2b_1} , q_{b_2b}) \leqq \underline{U}$.

(14) means that $mp_{a''a'}rkq_{b_2b_1} : Y_{b_2} \longrightarrow Y_{b'}$ satisfies the condition of Definition 4 for $\underline{q}$. Hence $\underline{q}$ is also approximatively movable. This completes the proof of Proposition 2.

By Proposition 2 we may define the following:

Definition 5. A space X is approximatively movable provided that any AP-resolution of X is approximatively movable.

Proposition 3. The notion of approximative movability is a topological invariant property.

This Proposition follows from the fact: Let $h : X \longrightarrow Y$ be a homeomorhism. Let $\underline{p} = \{p_a : a \in A\} \; Y \longrightarrow \underline{Y}$ be an AP-resolution of Y. Then $\underline{ph} = \{p_a h : a \in A\} : X \longrightarrow \underline{Y}$ forms an AP-resolution of X.

Proposition 4. Every approximative polyhedron is approximatively movable.

This Proposition follows from the fact: The rudimentary AP-resolution of an approximative polyhedron satisfies the condition of Definition 4.

4. Approximative movability and aproximatively internal movability for compact metric spaces. In this section we show the following relations among them.

Theorem 2. Let X be a compact metric space. Then X is approximatively movable if and only if it is approximatively internally movable.

By Theorems 1 and 2 we have that

Corollary 1. Let X be a compact metric space. Then X is an approximative polyhedron if and only if it is approximatively movable.

Proof of Theorem 2. First we assume that X is approximatively internally movable. Then X is an approximative polyhedron by Theorem 1. Therefore, by Proposition 4 it is approximatively movable.

Next, we assume that X is approximatively movable. Since X is compact metric, we may assume that X is a closed subset of the Hilbert cube Q. Let $\{V_i, p_{ij}, N\} = \underline{X}$ be an inverse sequence of compact ANRs in Q such that $V_i \geqq V_{i+1}$ and $p_{ij}: V_i \longrightarrow V_j$ are inclusion maps for each i and for $i \geq j$ with $\bigcap_{i=1}^{\infty} V_i = X$. Let $p_i : X \rightarrow V_i$ be the incusion map for each i. Then $\underline{p} = \{p_i\} : X \longrightarrow \underline{X}$ forms an inverse limit. Hence by Theorem 8 of Mardešić [4] $\underline{p}$ forms an AP-resolution of X.

We show that $\underline{p}$ is approximatively internally movable. Since X is approximatively movable, without of generality, we may assume that for each $i \geq 2$, there exist maps $r_i : V_i \rightarrow V_{i+1}$ such that

(1) $d(r_i(y), y) \leqq (1/2)^{i-1}$ for each $y \in V_i$.

Here d means a metric on Q. Take any $n \in N$ and any real number $t > 0$. Let n_0 be an integer such that $n_0 > n$ and $(1/2)^{n_0} < t/4$. Put $h_j = r_j r_{j-1} \cdots r_{n_0+1} : V_{n_0+1} \rightarrow V_{j+1}$ for each $j \geq n_0+1$. From (1) we have that for $j \geq k \geq n_0+1$, and for each $y \in V_{n_0+1}$,

(2) $d(h_j(y), h_k(y)) \leqq d(h_j(y), h_{j-1}(y)) + d(h_{j-1}(y), h_{j-2}(y)) + \ldots$

$\ldots, + d(h_{k+1}(y), h_k(y)) \leqq (1/2)^{j-1} + (1/2)^{j-2} + \ldots + (1/2)^k < (1/2)^{k-1}$.

(2) means that $\{h_j : j \geq n_0+1\}$ forms a Cauchy sequence, and hence $h = \lim \{h_j : j \geq n_0+1\} : V_{n_0+1} \longrightarrow Q$ is a continous map. However, since $h_k(y) \in V_{k+1}$ for each k, $h(y) = \lim h_k(y) \in V_{k+1}$ for each k, that is, $h(y) \in \bigcap_{k=1}^{\infty} V_k = X$ for each $y \in V_{n_0+1}$. Then $h : V_{n_0+1} \rightarrow X$.

From (1) we have that

(3) $d(h_j(y),y) \leqq d(h_j(y),h_{j-1}(y)) + d(h_{j-1}(y),h_{j-2}(y)) + \ldots$

$\ldots + d(h_{n_0+1}(y),y) \leqq (1/2)^{j-1} + (1/2)^{j-2} + \ldots + (1/2)^{n_0} < (1/2)^{n_0-1}.$

Since $(1/2)^{n_0} < t/4$, from (3) we have that

(4) $d(h(y),y) < t$ for each $y \in V_{n_0+1}$.

That is,

(5) $d(p_n h(y), p_{n_0+1,n}(y)) < t$ for each $y \in V_{n_0+1}$.

(5) means that p satisfies the condition of Definition 2, that is, p is approximatively internally movable. Hence X is approximatively internally movable. This completes the proof of Theorem 2.

We state the following Proposition without proof (see [5]).

Proposition 5. There exists a compact space X which is approximatively movable, but not approximatively internally movable.

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