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# On 2-normed spaces

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First we shall recall some definitions needed in the squel.

Let $X$ be a set. Consider a mapping $\rho : X \times X \times X \to R$.

$\rho$ is called a *2-metric* (S. Gähler), if it satisfies

(1) $\rho(x, y, z) \neq 0$ for any $x, y (x \neq y)$ and some $z$,

(2) $\rho(x, y, z) = 0$, if at least two points of therr points $x$, $y$, $z$ are equal,

(3) $\rho$ is symmetric on $x$, $y$, $z$, i. e. $\rho(x, y, z) = \rho(x, y, z) = \dots = \rho(z, y, x)$,

(4) $\rho(x, y, z) \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$.

By a *(linear) 2-normed space* $X$ *over reals* (S. Gähler), we mean a linear space $X$ in which to each pair of points $x, y \in X$, there exists a real number $||x, y||$ satisfying the following properties

(1) $||x, y|| = 0 \Leftrightarrow x, y$ are linear dependent,

(2) $||x, y|| = ||y, x||$,

(3) $||\alpha x, y|| = |\alpha| \, ||x, y||$ for any real $\alpha$,

(4) $||x, y + z|| \leq ||x, y|| + ||x, z||$.

We assume $\dim L \geq 2$.

On a 2-normed space $X$,

$$\rho(x, y, z) = ||y - x, z - x||$$

defines a 2-metric on $X$.

A *2-inner product space* (R. Ehret) is a linear space $X$ with a mapping $(\cdot, \cdot|\cdot) : X \times X \times X \to R$ satisfying the following conditions:

(1) $(a, a|b) \geq 0$, and $(a, a|b) = 0 \Leftrightarrow a, b$ are linearly dependent,

(2) $(a, a|b) = (b, a|a)$,

(3) $(a, b|c) = (b, a|c)$,

(4) $(\alpha a, b|c) = \alpha(a, b|c)$ for any real $\alpha$,

(5) $(a + a', b|c) = (a, b|c) + (a', b|c)$.

Proposition 1. On a 2-inner product space,

$$||a, b|| = \sqrt{(a, a|b)}$$

define a 2-norm for which

$$(a, b|c) = \frac{1}{4}(||a + b, c||^2 - ||a - b, c||^2)$$

and

$$||a + b, c||^2 + ||a - b, c||^2 = 2(||a, c||^2 + ||b, c||^2).$$

Proposition 2. Let $X$ be a pre-Hilbert space. Then

$$(a, b|c) = \begin{vmatrix} (a, b) & (a, c) \\ (b, c) & (c, c) \end{vmatrix}$$

$$= (a, b)||c||^2 - (a, c)(b, c)$$

defines a 2-inner product on $X$.

2-normed space $X$ is called *strictly convex* if for $a, b = 0$, $||a + b, c|| = ||a, c|| + ||b, c||$ and $||a, c|| = ||b, c|| = 1$, where $c$ is linearly independent to $a, b$, implies $a = b$, equivalently $||a, c|| = ||b, c|| = \frac{1}{2}||a + b, c|| = 1$, where $c$ is linear independent to $a, b$, implies $a = b$.

A 2-normed space $X$ is said to be *strictly 2-convex*, if $||a, b|| = ||a, c|| = ||b, c|| = \frac{1}{3}||a + c, b + c|| = 1$ implies $c = a + b$.

Let $c$ be a fixed non-zero element of a 2-normed space $X$, and let $V(c)$ be the linear subspace of $X$ generated by $c$.

Then we obtain the quotient space $X/V(c) = L_c$. We put

$$\|x\|_c = \|x, c\|,$$

then $\|\cdot\|_c$ is well-defined on $X_c$.

Proposition 3. $\|\cdot\|_c$ is a norm on $X_c$.

Proposition 4. A 2-normed space $X$ is strictly convex if and only if $X_c (c \neq 0)$ is strictly convex in usual sense, i. e. $\|x + y\|_c = \|X\|_c + \|y\|_c$, $\|X\|_c = \|y\|_c = 1$ imply $x = y$.

Some new characterizations of the strictly convexity by bounded linear 2-functionals in some sense are recently given by Y. Cho, K. Ha and W, Kim [ 1 ].

Quite recently, a wonderful characterization of a 2-inner product space is given by C. Diminnie and A. White [ 2 ].

Proposition 5. 2-normed space is a 2-inner product space if and only if

$$\begin{aligned} &\|x + y, y + z\|^2 + \|x + y, y - z\|^2 \\ &+ \|x - y, y + z\|^2 + \|x - y, y - z\|^2 \\ &= \|x, y\|^2 + \|x, z\|^2 + \|y, z\|^2 \end{aligned}$$

holds.

Next we concern with some special classes of mappings.

A 2-metric space $X$ is called to be complete, if for any sequence $\{x_n\}$, $\rho(x_m, x_n, a) \to 0$ for all $a \in X (m, n \to \infty)$ implies $\rho(x_m, x, a) \to 0$ for all $a \in X$ and for some $x \in X (m \to \infty)$.

The following fixed point theorem is obtained.

Proposition 6. Let $X$ be a bounded complete 2-metric space, and let $f_n$ be a sequence of mappings of $X$ into itself. If there are non-negative $\alpha, \beta$ such that for all $x, y \in X$

$$\begin{aligned} \rho(f_m(x), f_n(y), a) &\leq \alpha(\rho(x, f_m(x), a) \\ &+ \rho(y, f_n(y), a)) + \beta\rho(x, y, a) \end{aligned}$$

with $2\alpha + \beta < 1$, then the sequence $\{f_n\}$ has a unique common fixed point (K. Iséki, P. L. Sharma and B. K. Sharma).

Let $E$ be a usual normed space, and let $X$ be a 2-normed space. Then the following three mappings are considered.

(1) $f_1 : E \to X$ satisfies

$$||f(x) - f(y)|| = ||x - y, c||$$

for some fixed $c \in X$.

(2) $f_2 : X \to E$ satisfies

$$||f(x) - f(y), c|| = ||x - y||$$

for some fixed $c \in X$.

(3) $f_3 : X \to X$ satisfies

$$||f(x) - f(y), c|| \leq ||x - y, c||$$

for some fixed $c \in X$.

The first two mappings $f_1$, $f_2$ are due to C. Diminnie and A. White. $f_3$ is discussed by the present author. The mapping $f_3$ is uniquely determined. Roughly speaking this type is of an affine mapping (Iséki-Diminnie-White).

## References

[1] Y. J. Cho, K. S. Ha and W. S. Kim, Strictly convex linear 2-normed spaces, Math. Japonica, 26(1981), 475-478.

[2] C. Diminnie and A. White, A characterization of 2-inner product spaces, to appear in Math. Japonica.