数理解析研究所講究録
第 447 巻 1981 年 84-94

On double gaussian sums and theta transformation formula

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Introduction

It is known that in the Theta Transformation Formula

$$\theta_{\sigma\cdot m}(\sigma.\tau) = k(\sigma)\underline{e}(\phi_m(\sigma)\det(\gamma\tau+\delta)^{\frac{1}{2}}\theta_m(\tau)\cdots\cdots\cdots\cdots(1)$$

$k^8(\sigma) = 1$ [I, P.176, P.182]. A formula for $k(\sigma)$ would be of values in the theory of modular forms. For the genus one theta group $\Gamma_1(1,2)$ we have the following formula

$$k^2(\sigma) = \begin{cases} i^{d-1} = i^{a-1} = (-1)^{\frac{1}{2}(d-1)} = (-1)^{\frac{1}{2}(a-1)} & \text{if } a,d \text{ are odd.} \\ i^c = -i^b & \text{if } b,c \text{ are odd.} \end{cases}$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ \Gamma_1(1,2)$. [II]

In this paper, a formula of $k(\sigma)$ for a special kind of element in genus two theta group $\Gamma_2(1,2)$ is obtained.

§1. Theta group of genus two $\Gamma_2(1,2)$.

Let $\Gamma_2(1) = \left\{ \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \varepsilon\ GL(4, \mathbb{Z}) \middle| \sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

where $\alpha, \beta, \gamma, \delta, 0, 1$ are all two by two inegral matrices (last two being zero and identity). $\Gamma_2(1)$ is the Siegel modular group of genus two. A matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $M(4, \mathbb{Z})$ belongs to $\Gamma_2(1)$ if and only if $\alpha{}^t\beta$, $\gamma{}^t\delta$ are symmetric and $\alpha{}^t\delta - \beta{}^t\gamma = 1$.

Let

$$\Gamma_2(1,2) = \left\{ \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_2(1) \;\middle|\; (\alpha^t\beta)_o \equiv (\gamma^t\beta)_o \equiv 0 \bmod 2 \right\}$$

where $\begin{pmatrix} x & y \\ z & u \end{pmatrix}_o = \begin{pmatrix} x \\ u \end{pmatrix}$. $\Gamma_2(1,2)$ is a subgroup of $\Gamma_2(1)$, usually called Theta group of genus 2. In this paper we shall consider only those σ in $\Gamma_2(1,2)$ whose γ is a scalar matrix. In below we shall state without proof some properties of $\Gamma_2(1,2)$ which we need in the sequel.

Lemma 1. Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \varepsilon\ \Gamma_2(1,2)$, $\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, c is a positive integer. Then

(i) δ is symmetric, hence $\delta = \begin{pmatrix} d_1 & d \\ d & d_2 \end{pmatrix}$. Similarily,

α is symmetric, hence $\alpha = \begin{pmatrix} a_1 & a \\ a & a_2 \end{pmatrix}$.

(ii) cd_1 and cd_2 are both even

(iii) $(c, d, d_1) = (c, d, d_2) = 1$.

(iv) $(c, \det \delta) = (c, \det \alpha) = 1$.

Lemma 2. (classification)

In the notation of Proposition 1, σ can be classified into the following four cases:

(I) c is odd; then d_1, d_2 are both even.

(II) c is even, d_1 and d_2 are both even; hence d is odd.

(III) c is even, d_1 and d_2 are of opposite parity; hence

d is odd. Without loss of generality, we may assume d_1 is odd and d_2 is even.

(IV) c is even, d_1 and d_2 are both odd; hence d even.

§2. Theta transformation formula

In [III, P.181] Krazer gave another form of theta transformation formula, for $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \varepsilon\ \Gamma_2(1)$, $\det \gamma \neq 0$,

$$\theta_m(\tau) = (\det \gamma)^{-\frac{1}{2}} \det(\gamma\tau+\delta)^{-\frac{1}{2}} (\pi i) |\det \gamma|^{-1} G \underline{e}(\tfrac{\phi}{2}) \underline{e}(\tfrac{\psi(m)}{2}) \theta_{\sigma\cdot m}(\sigma\cdot\tau) \quad \cdots\cdots (2)$$

Equating formula (1) and (2), then put $m = 0$, we get

$$k(\sigma) = -i\ |\det \gamma|^{\frac{3}{2}} G^{-1}\ \underline{e}(-\tfrac{\phi}{2}) \quad \cdots\cdots (3)$$

where

$$\begin{cases} \phi = -\dfrac{1}{4}\ {}^t(\gamma^t\delta)_0 \alpha\gamma^{-1}(\gamma^t\delta)_0 - \dfrac{1}{2}\ {}^t(\gamma^t\delta)_0(\alpha^t\beta)_0 & \cdots\cdots (4) \\ G = \displaystyle\sum_{\xi_1,\xi_2=0}^{|\det\gamma|-1} \underline{e}(-\tfrac{1}{2}\ \xi(\gamma^{-1}\delta)^t\xi + \tfrac{1}{2}\ \xi\gamma^{-1}(\gamma^t\delta)_0) & \cdots\cdots (5) \end{cases}$$

Lemma 3. If $\sigma\ \varepsilon\ \Gamma_2(1,2)$, $\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, c is a positive integer, then

$$\underline{e}(\tfrac{-\phi}{2}) = \begin{cases} 1 & \text{for case (I) and (II)} \\ (-1)^{(c/2)(a_1/2)} & \text{for case (III)} \quad \cdots\cdots (6) \\ (-1)^{(c/2)(a_1+a_2)/2} & \text{for case (IV)} \end{cases}$$

§3. Degree two gaussian density G.

Lemma 4. With notations as in Lemma 3, G defined by (5), we have

$$G = c^2 \sum_{\text{mod } c} \underline{e}(-\tfrac{1}{2c}\ \xi\delta^t\xi + \tfrac{1}{2}\ \xi(\delta)_0)$$

Let $H = H(c,\delta) = \sum_{\xi \bmod c} \underline{e}(-\frac{1}{2c}\xi\delta^t\xi + \frac{1}{2}\xi(\delta)_o)$ and let $\psi(n,T) = \sum_{\xi \bmod n} \underline{e}(\frac{1}{n}\xi T^t\xi)$ where n is a positive integer and T is an two by two integer symmetric matrix.

Lemma 5.

$$H = \begin{cases} \frac{1}{4}\ \psi(2c, -\delta) & \text{for case (I), } d \text{ is odd} \\ \psi(c, -\frac{\delta}{2}) & \text{for case (I), } d \text{ is even} \\ \frac{1}{4}\psi(2c, -\delta) & \text{for case (II)} \\ \psi(c, -\delta') - \frac{1}{4}\psi(2c, -\delta) & \text{for case (III)} \\ H(c,\delta'') & \text{for case (IV)} \end{cases}$$

where

$$\delta' = \begin{pmatrix} 2d_1 & d \\ d & \frac{d_2}{2} \end{pmatrix} \qquad \delta'' = \begin{pmatrix} d_1 & d_1+d \\ d_1+d & d_1+d_2+2d \end{pmatrix}.$$

§4. Separation and reduction of double gaussian sums $\psi(n,T)$.

Double gaussian sums $\psi(n,T)$ was considered by Weber in [IV], the following three propositions are keys for the computation of $\psi(n,T)$:

Lemma 6. [IV, P.36] If $n > 0$ is an odd integer, and if $(\det T,n)=1$, then

$$\psi(n,T) = (\frac{\det T}{n})(-1)^{\frac{n-1}{2}}\, n$$

where (-) is the Legendre symbol.

Lemma 7. (Separation) [IV, P.17] If $(n_1,n_2) = 1$ then

$$\psi(n_1n_2, T) = \psi(n_1, n_2T)\psi(n_2, n_1T).$$

Lemma 8. (Reduction) [IV, P.44]

If $\det T$ is odd, then $\psi(2^p,T) = 4\,\psi(2^{p-2},T)$ if $p \geq 4$ or if $p \geq 3$ and $T_o \equiv 0 \mod 2$.

Lemma 9. Let T be an integral symmetric matrix,

then
$$\psi(1,T) = 1$$
$$\psi(2,T) = [1 + (-1)^{t_1}] \times [1 + (-1)^{t_2}]$$
$$\psi(2^2,T) = 2^2(1 + i^{t_1} + i^{t_2} + i^{t_1+t_2+2t})$$
$$\psi(2^3,T) = 2^4 \cdot i^{t_2/2} \quad \text{if } t_1, t \text{ odd, } t_2 \text{ even.}$$

It follows from Lemma 9 that

$$\psi(2,T) = \begin{cases} 2^2 & \text{if } (T)_o \equiv 0 \mod 2. \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(2^2,T) = \begin{cases} 2^3\ (-1)^{(t_1/2)\cdot(t_2/2)} & \text{if } t_1 \equiv t_2 \equiv 0,\ t \not\equiv 0 \mod 2 \\ 2^3 & \text{if } t_1 \equiv t \equiv 1 \mod 2,\ t_2 \equiv 0 \mod 4. \\ 2^3 \cdot i^{-t_1} & \text{if } t_1 \equiv t \equiv 0 \mod \equiv 2,\ t_2 \equiv 2 \mod 4. \end{cases}$$

§5. Calculations and final results

(Case I) If d is even then $H = \psi(c, -\frac{\delta}{2}) = \left(\frac{\det(-\frac{\delta}{2})}{c}\right) \cdot (-1)^{\frac{c-1}{2}} c = \left(\frac{\det \delta}{c}\right)(-1)^{\frac{c-1}{2}} c$, by Lemmas 5 and 6 and the fact $\left(\frac{\det \delta}{c}\right) = \left(\frac{2^2 \det(\delta/2)}{c}\right) = \left(\frac{\det(\delta/2)}{c}\right) = \left(\frac{\det(-\delta/2)}{c}\right)$.

If d is odd, $H = \frac{1}{4}\,\psi(2c, -\delta) = \frac{1}{4}\,\psi(c, -2\delta)\psi(2, -c\delta)$ $= \frac{1}{4}\left(\frac{\det(-2\delta)}{c}\right)(-1)^{(c-1)/2} \cdot c \times 4 = \left(\frac{\det \delta}{c}\right)(-1)^{(c-1)/2} c$, by Lemma 5 and 6 and the fact that

$$\psi(2, -c\delta) = \sum_{\xi \bmod 2} e\left(\frac{-c}{2}\, \xi \delta {}^t\xi\right) = 4.$$

Therefore, in case (I), $H = \left(\frac{\det \delta}{c}\right)(-1)^{(c-1)/2}c$.

(Case II)
Let $c = 2^{p-1}\cdot q$

$$H = \frac{1}{4}\psi(2c, -\delta)$$

$$= \frac{1}{4}\psi(2^p q, -\delta)$$

$$= \frac{1}{4}\psi(2^p, -q\delta)\psi(q, -2^p\delta)$$

$$= \frac{1}{4}\left(\frac{\det(-2^p\delta)}{q}\right)(-1)^{(q-1)/2}\, q \times \psi(2^p, -q\delta)$$

$$= \frac{1}{4}\left(\frac{\det \delta}{q}\right)(-1)^{(q-1)/2}\, q \times \psi(2^p, -q\delta) \cdots\cdots\cdots\cdots (6)$$

in which $p \geq 2$ for c is even.

Suppose p is odd, $p = 2k+1$, then by reduction k times and Lemma 9,

$$\begin{aligned}\psi(2^p, -q\delta) &= \psi(2^{2k+1}, -q\delta) \\ &= 4^k\psi(2, -q\delta) \\ &= 4^k \times 4 \\ &= 4^{k+1} \\ &= 2^{p+1}. \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)\end{aligned}$$

Suppose p is even, $p = 2k$, then by reduction $k-1$ times and Lemma 9,

$$\begin{aligned}\psi(2^p, -q\delta) &= \psi(2^{2k}, -q\delta) \\ &= 4^{k-1}\psi(2^2, -q\delta) \\ &= 4^{k-1} \times 2^3 \times (-1)^{(d_1/2)\cdot(d_2/2)} \\ &= 2^{2k+1}\cdot(-1)^{(d_1/2)\cdot(d_2/2)}\end{aligned}$$

$$= 2^{p-1}\cdot(-1)^{(d_1/2)\cdot(d_2/2)} \qquad \text{.................. (8)}$$

(7) and (8) can be combined in one formula:

$$\psi(2^p, -q\delta) = 2^{p+1}\cdot(-1)^{(p-1)\cdot(d_1/2)\cdot(d_2/2)} \qquad \text{........... (9)}$$

Put (9) into (6), we get

$$H = \left(\frac{\det\ \delta}{q}\right)(-1)^{(q-1)/2}\cdot q\cdot 2^{p-1}\cdot(-1)^{(p-1)\cdot(d_1/2)\cdot(d_2/2)}$$

$$= \left(\frac{\det\ \delta}{q}\right)(-1)^{(q-1)/2\ +\ (p-1)\cdot(d_1/2)\cdot(d_2/2)}\cdot c\ .$$

(Case III)

$$H = \psi(c, -\delta') - \frac{1}{4}\psi(2c, -\delta) \qquad \text{.......................... (10)}$$

$$\psi(2c, -\delta) = \psi(2^p q, -\delta)$$

$$= \psi(q, -2^p\delta)\ (2^p, -q\delta) \qquad \text{...................... (11)}$$

where $\psi(q, -2^p\delta) = \left(\frac{\det\ \delta}{q}\right)(-1)^{(q-1)/2}q$ by proposition 6

Suppose p is odd, $p = 2k+1$, then by reduction $k-1$ times and Lemma 9

$$\psi(2^p, -q\delta) = \psi(2^{2k+1}, -q\delta)$$

$$= 4^{k-1}\psi(2^3, -q\delta)$$

$$= 4^{k+1}i^{-q(d_2/2)}$$

$$= 2^{p+1}\ i^{-q(d_2/2)} \qquad \text{........................... (12)}$$

Suppose p is even, $p = 2k$, then by reduction $k-1$ times and Lemma 9,

$$\psi(2^p, -q\delta) = \psi(2^{2k}, -q\delta)$$

$$= 4^{k-1}\psi(2^2, -q\delta)$$

$$= 2^{p+1}\times\begin{cases} 1 & \text{if } d_2/2 \text{ even} \\ i^{-2d_1} & \text{if } d_2/2 \text{ odd,} \end{cases} \qquad \text{......... (13)}$$

Put (12), (13) into (11), we get

$$\psi(2c,-\delta) = \left(\frac{\det\ \delta}{q}\right)(-1)^{(q-1)/2}\ q \times 2^{p+1} \times \begin{cases} i^{-q(d_2/2)} & \text{if } p \text{ is odd} \\ 1 & \text{if } p,\ d_2/2 \text{ are even} \\ i^{-2d_1} & \text{if } p \text{ is even, } d_2/2 \text{ is odd.} \end{cases}$$

$$= 4\left(\frac{\det\ \delta}{q}\right)(-1)^{(q-1)/2}\ c \times \begin{cases} i^{-q(d_2/2)} & p \text{ odd} \\ 1 & p,\ d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even, } d_2/2 \text{ odd.} \end{cases} \quad \cdots\cdots(14)$$

On the other hand,

$$\psi(c,-\delta') = \psi(2^{p-1}\cdot q,\ -\delta')$$
$$= \psi(q,-2^{p-1}\delta')\psi(2^{p-1},-q\delta') \cdots\cdots\cdots\cdots\cdots\cdots(15)$$

where $\psi(q,-2^{p-1}\delta') = \left(\frac{\det\ \delta}{q}\right)(-1)^{(q-1)/2}\ q$ by Lemma 6 and the fact $\det\ \delta' = \det\ \delta$.

Suppose p is odd, $p = 2k+1$, then by reduction $k-1$ time and Lemma 9, we get

$$\psi(2^{p-1},-q\delta') = 4^{k-1}\psi(2^2,-q\delta')$$
$$= 2^p\ i^{-(d_2/2)\cdot q} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(16)$$

Suppose p is even, and $p \geq 4$, $p = 2k$, then by reduction $k-2$ times and Lemma 9, we get

$$\psi(2^{p-1},-q\delta') = 4^{k-2}\psi(2^3,-q\delta')$$
$$= 2^p \times \begin{cases} 1 & \text{if } d_2/2 \text{ even} \\ i^{-qd_1} & \text{if } d_2/2 \text{ odd.} \end{cases} \quad \cdots\cdots\cdots\cdots(17)$$

Put (16), (17) into (15), we get

$$\psi(c,-\delta') = \left(\frac{\det\,\delta}{q}\right)(-1)^{(q-1)/2}\cdot q \times 2^p \times \begin{cases} i^{-(d_2/2)\,q} & p \text{ odd} \\ 1 & p,\ d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even},\ d_2/2 \text{ odd.} \end{cases}$$

$$= 2\left(\frac{\det\,\delta}{q}\right)(-1)^{(q-1)/2}\cdot c \times \begin{cases} i^{-(d_2/2)\,q} & p \text{ odd} \\ 1 & p,\ d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even},\ d_2/2 \text{ odd.} \end{cases} \qquad \cdots\cdots\cdots\cdots(18)$$

Put (18) and (14) into (10), we get

$$H = \left(\frac{\det\,\delta}{q}\right)(-1)^{(q-1)/2}\, c \times \begin{cases} i^{(d_2/2)\cdot q} & p \text{ odd} \\ 1 & p,\ d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even},\ d_2/2 \text{ odd} \end{cases}$$

In case $p = 2$, i.e $c = 2q$, we get from Lemma 9 that

$$\psi(2,\ -q\delta) = \begin{cases} 4 & \text{if } d_2/2 \text{ is even} \\ 0 & \text{otherwise} \end{cases} \qquad \cdots\cdots\cdots\cdots\cdots(19)$$

Put (19) into (15), we get

$$\psi(c,-\delta') = 2\left(\frac{\det\,\delta}{q}\right)(-1)^{(q-1)/2}\, c \times \begin{cases} 1 & d_2/2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Put (20) and (14) into (10), we get

$$H = \left(\frac{\det \delta}{q}\right)(-1)^{(q-1)/2}\, c \times \begin{cases} 2 - 1 = 1 & \text{if } p=2,\ d_2/2 \text{ even} \\ 0 - i^{-qd_1} = -i^{-qd_1} & \text{if } p=2,\ d_2/2 \text{ odd} \end{cases}$$

(Case IV):

By Lemma 5, $G = G(c,\delta) = G(c,\delta'')$ where $\delta'' = \begin{pmatrix} d_1 & d+d_1 \\ d+d_1 & d_1+d_2+d \end{pmatrix}$ belongs to case III, hence Case IV follows from case III.

In summary, we have

<u>Proposition 1</u>. If $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \varepsilon\ \Gamma_2(1,2)$, $\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, $\delta = \begin{pmatrix} d_1 & d \\ d & d_2 \end{pmatrix}$, $c > 0$, $c = 2^{p-1}\cdot q$, q is odd then

$$G = \begin{cases} c^3\left(\frac{\det}{c}\right)(-1)^{(c-1)/2}. & \text{if } c \text{ is odd} \\ c^3\cdot\left(\frac{\det}{q}\right)(-1)^{(q-1)/2 + (d_1/2)\cdot(d_2/2)(p-1)} & \text{if } c,\ d_1, d_2 \text{ are all even} \\ c^3\cdot\left(\frac{\det}{q}\right)(-1)^{(q-1)/2}\cdot\ \zeta & \text{if } c \text{ is even, } d_1 \text{ is even and } d_2 \text{ is odd.} \\ c^3\cdot\left(\frac{\det}{q}\right)(-1)^{(q-1)/2}\cdot\ \xi & \text{if } c \text{ is even, } d_1 \text{ and } d_2 \text{ are odd.} \end{cases}$$

where

$$\zeta = \begin{cases} i^{-(d_2/2)q} & p\text{:odd} \\ 1 & p \text{ and } \frac{d_2}{2} \text{ even} \\ i^{-qd_1} & p > 2 \text{ even, } d_2/2 \text{ odd} \\ -i^{-qd_1} & p = 2,\ d_2/2 \text{ odd.} \end{cases}$$

$$\xi = \begin{cases} i^{-[(d_1+d_2+2d)/2]q} & \text{p: odd} \\ 1 & \text{p: even and } d_1 \not\equiv d_2 \bmod 4 \\ i^{-qd_1} & \text{p>2, even, and } d_1 \equiv d_2 \bmod 4 \\ -i^{-qd_1} & \text{p=2, } d_1 \equiv d_2 \bmod 4 \end{cases}$$

Proposition 2. Notations as in Proposition 1, we have correspondingly

$$K(\sigma) = \begin{cases} i\left(\frac{\det \delta}{c}\right)(-1)^{(c+1)/2} \\ i\left(\frac{\det \delta}{q}\right)(-1)^{(q+1)/2 + (d_1/2)(d_2/2)(p-1)} \\ i\left(\frac{\det \delta}{q}\right)(-1)^{(q+1)/2 + (c/2)(a_1/2)} \cdot \xi^{-1} \\ i\left(\frac{\det \delta}{q}\right)(-1)^{(q+1)/2 + (c/2)(a_1+a_2)/2} \cdot \xi^{-1} \end{cases}$$

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