SOME PROBLEMS IN NUMBER THEORY

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We give below a selection of problems on Dirichlet series, modular forms and transcendental numbers.

Dirichlet series

Let $s = \sigma + it$, $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$, $\lambda_{n+1} - \lambda_n \gg$ and $\ll 1$, put

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\lambda_n^{s}} , \ (\sigma > 0).$$

Problem 1 Show that $f(s) = 0$, $\sigma \geq \frac{1}{2}$ has an infinity of solutions.

Problem 2 Show that $|f(\frac{3}{4} + it)| = \Omega((\text{Exp}(\log t)^{1/8})), \ t \geq 10$.

Transcendental numbers

Prove that the number of algebraic numbers in $2^\pi, 2^{\pi^2}, \ldots, 2^{\pi^N}$ is $\mathcal{O}(N^\frac{1}{\pi})$.

$\mathcal{O}(N^\frac{1}{\pi})$ has been proved by K. Ramachandra and S. Srinivasan. This will appear in Hardy Ramanujan Journal, Vol. 6 (1983).

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a cusp form of weight $k$ for the full modular group $\text{SL}_2(\mathbb{Z})$. Define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

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and
\[ L_2(s,f) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}. \]

Then it is well-known that \( L(s,f) \) has an analytic continuation to the entire complex plane. The function \( \zeta(2s)L_2(s + k - 1,f) \) also has an analytic continuation to the entire plane except for a simple pole at \( s = 1 \).

1. Show that if \( f \) is a normalized Hecke eigenform, then
\[ (2\pi)^{-s} \Gamma(s)L(s,f) \]
is a monotone increasing function for \( s \geq \frac{k}{2} \).

2. Give an example of a cusp form such that \( L_2(s,f) \) has a zero for some \( s \) satisfying \( \text{Re}(s) > k - \frac{1}{2} \).

3. Show that if \( f \) is a normalized Hecke eigenform, then \( L_2(s,f) \) has no real zero \( s \) satisfying \( k - \frac{1}{2} < \text{Re}(s) < k \).