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SOME PROBLEMS IN NUMBER THEORY

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We give below a selection of problems on Dirichlet series, modular forms and transcendental numbers.

Dirichlet series

Let $s = \sigma + it$, $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$,

$\lambda_{n+1} - \lambda_n \gg$ and $\ll 1$, put $f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\lambda_n^s}$, $(\sigma > 0)$.

Problem 1 Show that $f(s) = 0$, $\sigma \geq \frac{1}{2}$ has an infinity of solutions.

Problem 2 Show that $|f(\frac{3}{4} + it)| = \Omega(\mathrm{Exp}((\log t)^{1/8}))$, $t \geq 10$.

Transcendental numbers

Prove that the number of algebraic numbers in

$2^{\pi}, 2^{\pi^2}, \ldots, 2^{\pi^N}$ is $o(N^{\frac{1}{2}})$.

$O(N^{\frac{1}{2}})$ has been proved by K. Ramachandra and S. Srinivasan. This will appear in Hardy Ramanujan Journal, Vol. 6 (1983).

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a cusp form of weight k for the full modular group $SL_2(\mathbb{Z})$. Define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$L_2(s,f) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s} .$$

Then it is well-known that $L(s,f)$ has an analytic continuation to the entire complex plane. The function $\zeta(2s)L_2(s + k - 1,f)$ also has an analytic continuation to the entire plane except for a simple pole at $s = 1$.

1. Show that if f is a normalized Hecke eigenform, then

$$(2\pi)^{-s}\Gamma(s)L(s,f)$$

is a monotone increasing function for $s \geq \frac{k}{2}$.

2. Give an example of a cusp form such that $L_2(s,f)$ has a zero for some s satisfying $\mathrm{Re}(s) > k - \frac{1}{2}$.

3. Show that if f is a normalized Hecke eigenform, then $L_2(s,f)$ has no real zero s satisfying $k - \frac{1}{2} < \mathrm{Re}(s) < k$.